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ABSTRACT

This book is designed for those interested in the teaching of mathematics, in both first language and second language contexts, and is based on 15 years' teaching experience in Africa and Asia. The book is designed to present the main issues of language in mathematics teaching, and is therefore not a highly technical work. Chapters included are: (1) "Introduction: Is Mathematics a Language?"; (2) "The Development of Formal Thought through Language: Piaget vs. Vygotsky"; (3) "Concepts" (discussing research on how the naming of concepts affects the learning of those concepts); (4) "Logical Connectives"; (5) "Problem Solving"; (6) "Correlational Studies" (describing the relationships between mathematical ability and language skills); (7) "The Whorf Hypothesis" (summarizing the results of hypothesis testing); (8) "Bilingual Education" (discussing the effects of bilingual education on mathematics learning); (9) "The Alphabet Effect" (discussing the effects of written language on logic and on learning); (10) "Sociolinguistics in the Classroom"; (11) "Ethnomathematics" (discussing the influence of cultural factors on mathematics and mathematics education); and (12) "Applications and Conclusions." (YP)

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Language and Mathematics Education

Raymond Zepp

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LANGUAGE
AND
MATHEMATICS EDUCATION

LANGUAGE AND MATHEMATICS EDUCATION

RAYMOND ZEPP

API Press

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THE AUTHOR

Born and educated in the United States, Raymond Zepp graduated from Oberlin College and received his PhD from Ohio State University, where he was Lecturer in Mathematics. He taught for eight years in African universities: in Nigeria, Lesotho, and finally in the Ivory Coast, as an American Fulbright Professor. There he set up vernacular numeracy courses for illiterate farmers of nine linguistic groups. He was USAID consultant to Haiti, where he worked in numeracy training for Creole-speaking adults. He has authored several books and numerous research articles, many on linguistic aspects of mathematical logic. He has taught for the past five years at the University of East Asia, Macau.

FOREWORD

I am delighted to write a short foreword for this important new contribution to the interaction of language with mathematical education. Raymond Zepp has brought together the research literature on a wide range of issues which are of the utmost importance for mathematics educators. In a very convincing way the author takes us on a journey which crosses many interesting landscapes. The borders between each type of country are not always clearly visible and as one merges into the other we find ourselves attracted to some new aspect of language which demands our attention. Fortunately our guide does not allow us to wander too far on sidetracks, attractive though many of them are. His main purpose is always before us and before we enter new territory he carefully considers the implications of what we have seen for classroom practice. The result is an intriguing experience. I am particularly impressed with the juxtaposition of pure mathematical concepts with linguistic issues. Here is a writer who takes the trouble to illustrate his arguments with good examples. This makes the writing very appealing to me as a teacher of mathematics. On the other hand, as a researcher, I find the text full of ideas for new research, desperately needed replications to substantiate a particular theory, and suggestions for serious, as yet unanswered questions in any of a dozen areas. A glance through the chapter headings shows how diverse these areas are. The literature review in each chapter is substantial and up to date, providing an excellent base from which to work.

In recent years there have been many publications which have sensitized the mathematics education community to the importance of language in teaching and learning mathematics. This is the most comprehensive attempt I have seen to "tie together the many cross-disciplinary loose ends". It is the author's intention that we come to the inevitable conclusion that as mathematics teachers we cannot escape being language teachers. Perhaps you are already convinced? Read on - this book provides guidelines for action. Are you unconvinced? Read on - this book will surprise you.

Lloyd Dawe
Sydney Institute of Education, Sydney, Australia

PREFACE

This book is designed for those interested in the teaching of mathematics, in both first language and second language contexts. This group includes principally mathematics teachers and educators in the second/foreign language field. It is designed to acquaint these people with the main issues of language in mathematics teaching, and is therefore not a highly technical work. An attempt has been made to avoid linguistic jargon where possible, and to keep the language simple for easy reading.

I myself come to this field from fifteen years of experience as a teacher of mathematics in Africa and Asia, where the medium of instruction has been English or French rather than the first language of the students. I have observed some of the difficulties of students learning mathematics in a foreign language in which they are not always very proficient. I have heard claims and counterclaims about the importance of language in mathematics education, and have become interested in investigating some of these claims for myself. I find it a fascinating field of investigation, one which dips into many academic disciplines: education, psychology, mathematics, sociology, and of course linguistics. It is perhaps presumptuous to attempt to write a book which encompasses all these areas of expertise, but I feel a start must be made somewhere at tying together the many cross-disciplinary loose ends. I have been greatly aided in the process by some interested and sympathetic linguists and educators, notably Mary Willes and Nigel Bruce, to whom I turned for advice on many occasions. In addition, the book owes its existence to Ms. Christine Swales, whose professional comments and suggestions were of inestimable value.

A book of this size cannot claim to be a comprehensive review of all the literature on the subject. However, the major contributions to the theory are discussed, and the reader is directed to the main sources of the literature for further details. The book is intended as an overview which introduces teachers to the idea that language does play a major role in the formation of mathematical concepts, in the use of formal logical inference, and in the social structure of the classroom.

This is not to say that language is all-important in mathematics learning. Some theories assert that language can account for almost everything. This book paints a mixed picture and shows instances where some of these theories are over-ambitious in their scope. I hope that it presents a balanced view which airs the strong and weak points of many of the theories. Still, there are several common threads to many of the theories, and some conclusions can be drawn. In any event, the evidence should point to the conclusion that mathematics education and language have many connections, and mathematics teachers cannot escape being language teachers as well.

R. A. Zepp

CHAPTER I

INTRODUCTION: IS MATHEMATICS A LANGUAGE?

The question asked by the title of this introductory chapter involves a rather philosophical question: "What is language?". This book does not attempt to be a philosophy text, but in this case a certain amount of philosophy is necessary. The chapter cannot attempt an all-inclusive definition of language, and without such a definition, the answer to the question in the chapter's title will remain partially unanswered.

In fact, even questions such as "Is American English a language?"; "Is Black American English a language?"; or "Is English one language?" are difficult to answer even by the most expert linguists, owing to the difficulty in defining 'language'. As a result, the number of languages in the world cannot even be estimated to the nearest ten thousand - estimates range from 30,000 to 60,000 - depending on the classification scheme. Consequently, even a trained linguist could not really say whether mathematics is a language or not.

To pursue further this notion of the classification of languages, consider as an example the Senoufo 'language' of West Africa. In one Senoufo village A, most people will understand everything people say, but in village B, only a few kilometres away, people may understand only 90% of the speech from village A; perhaps 100 kilometres away, in village C, people will understand, say, 50% of the speech from village A; and in village D, 500 kilometres away, people will understand only a few words of speech A. What emerges is not a set of languages, but a continuum of language which changes more or less continuously. Who can say where one language stops and another begins?

Linguists have been able to measure the amount of inter-comprehensibility between two groups; that is, they can put a number on the percent of speech A understood by a speaker of a slightly different 'language' B. The following map of the Senoufo region, developed by Mills (cited in Zepp, 1983) depicts the

percentages of the central Korhogo dialect understood by peoples from other areas.

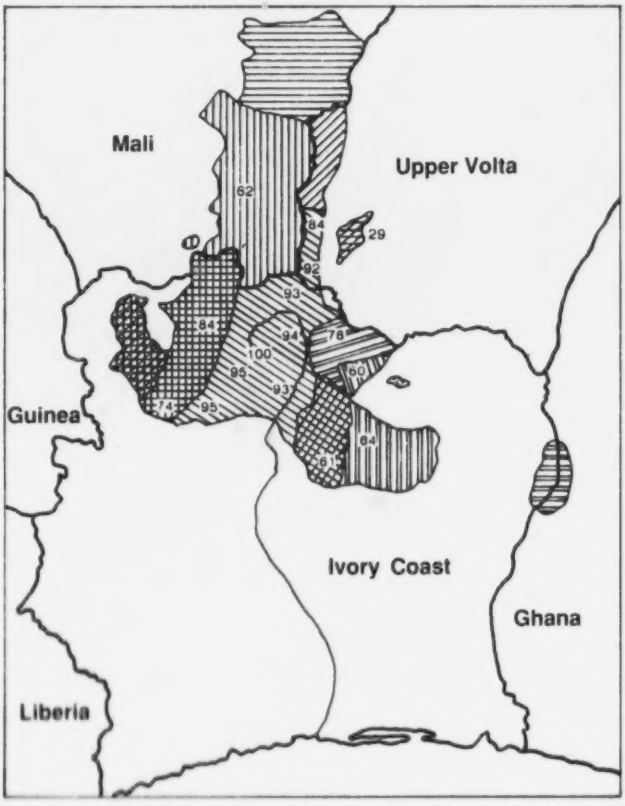

Figure 1.1 Levels of comprehension of Senoufo dialects

I have dwelt on this example in order to make an important point: our traditional, set theoretical mathematics is not able to define a 'set' of speakers of the Korhogo Senoufo language. It is impossible to state precisely whether or not a person from, say, Banfora, is a speaker of the Korhogo language. Mathematicians' preoccupation with well-defined, black-and-white sets, can limit their ability to analyze situations where a yes-or-no answer is impossible. The question, "Is mathematics a language?" is typical

of mathematicians, as it expects a yes-or-no answer, as if there were a well-defined 'set' of languages. A linguist, or perhaps a member of another culture, might never even think of the question.

The Senoufo 'language' is a textbook example of a 'fuzzy set', in which an element can be a partial member. The theory of fuzzy sets is systematically described in McCawley (1981). Each element of the underlying universal set has a number which describes 'how much' the element belongs to the set in question. A Senoufo from Banfora, who understands 29% of Korhogo speech, can be said to be a 29% member of the fuzzy set of Korhogo speakers. In a traditional set, unlike a fuzzy set, an object is either 100% member or 0% member.

This is discussed in Chapter 3, which explores the way the human mind classifies things, but for now it suffices to emphasize that traditional mathematics, with its set-theoretical bias, represents just one way of looking at the world, one which may not always be the most appropriate for analyzing a situation, one which may not be psychologically natural for most people, and one which may be very foreign to peoples of non-Western cultures.

The question, "Is mathematics a language?" therefore presupposes some possibly false ideas about the zero-one nature of languages, and to answer it is similar to attempting to answer the question "When did you stop beating your wife?", which also presupposes a possible falsehood. The aim of this chapter is rather to discuss how we look at language, how we learn it, and how we use it. The focus will be on the question of whether we look at, learn, and use mathematics in the same way as language.

Another point concerning the original question must be clarified. Does the word 'mathematics' refer to mathematical language or to all of mathematics itself? If the former is the case, the question becomes: "Is the language used to perform and describe mathematics a separate language in its own right, distinct from, say, English?" If the latter is the case, it will be necessary to examine a precise definition of just what mathematics is.

If it is mathematical language which is in question, we could ask the same question about any other discipline of study, such as sociology, physics, law or music. Is the language of sociology a language of its own, as distinct from English? Most people would probably argue that sociology uses a complex vocabulary and could be classified as jargon, or some specialized subset of English, but is not a separate language. Even if it is

treated in the fuzzy sense, similar to a dialect of the Senoufo language, it might be possible for the layman to understand a certain percentage of a sociological discourse, and so sociological language could be said to be partially English.

But to many students the comprehension of a mathematical statement seems to require something more than the comprehension of a sociological one. There is a different *way* of thinking involved, which makes mathematics appear to many people as something totally foreign. It is that foreignness of thought which makes people regard mathematics as a foreign language and not just a subset of whatever language it appears to be written in.

So perhaps it is mathematical thought itself of which the question must be asked, "Is mathematical thought different from thought in our own everyday language?" Here we address that question, to which there is no easy answer. We describe the attempts of many theorists to answer it, and there are differing opinions over the answer. The crux of the matter, of course, is the question of the relationship of thought to language. Is all thought performed in language? Or do thoughts exist independently from language, only later to be encoded into language? The best answer possible today is that thought is not identical to language, but that abstract mathematical thought cannot exist in isolation from language. Theorems must be proved, and although theorems cannot be said to be simply collections of words, it would be impossible to prove the theorem without words. There would be no way for an illiterate deaf-mute to grasp or to prove the fundamental theorem of calculus.

Mathematics is defined simply in the *Oxford Advanced Learner's Dictionary* (Hornby, 1987) as "the science of size and numbers", and so the question of whether mathematics is language comes down to whether science is language. Science is defined as "knowledge arranged in an orderly manner, especially knowledge obtained by observation and testing of facts; pursuit of such knowledge." Surely language also is structured knowledge, and is obtained by experience. But is the "pursuit of such knowledge" also language? This question has at its base the distinction between knowledge and the method of obtaining it. The definition claims that science is knowledge which has been acquired in a certain way. This claim presupposes that the structure of knowledge may depend on the way it is acquired.

Two of the principal ways of approaching intellectual questions are the structural approach and the functional approach.

The structural approach takes the position, "If you want to know something about an object, you must study the object itself." The functional approach, on the other hand, looks at the object in its context and asks, "What does the object do?" It is possible to study language in both ways. The 1950s and 1960s were marked by a structural bias, while the past two decades have seen a rise in the functionalist method of analysis, not just in linguistics, but in most of the humanities and the social sciences.

Language and mathematics as structured systems - Chomsky

Linguistics during the 1960s was dominated by Noam Chomsky, who was very much interested in the structure of language. He himself rejected the term 'structuralist' as a description of his work, since for him the word implied a study of only surface structure. In so stating, he criticized leading structural linguists of the turn of the century such as Ferdinand de Saussure (1916), or W. D. Whitney, whom Chomsky quotes (without reference) as stating, "Language in the concrete sense [is] the sum of words and phrases by which man expresses his thought." Such a view, according to Chomsky (1968), treated only the surface structure of language, while ignoring its 'deep structure', a term for which Chomsky became famous. He described 'deep structure' as follows:

> Surface structure corresponds only to sound - to the corporeal aspect of language; but when the signal is produced, with its surface structure, there takes place a corresponding mental analysis into what we may call the deep structure, a formal structure that relates directly not to the sound but to the meaning. (p. 65)

Now the surface structure and deep structure of language can be compared with mathematical equations and inequalities, which follow certain grammatical rules. We may state whether an equation is correct or incorrect grammatically (as opposed to true or false). A mathematician would consider the inequality

$$2 < 3 > 1$$

is grammatically incorrect, so that we cannot even say whether it is true or false.

But the grammatical rules are simply evidence of surface structure. The two equations:

$$3x + 4 = 7 \quad , \quad \text{and}$$
$$x = 1$$

are obviously different equations on the surface. But they share a deeper structure in that they have the same 'meaning' or the same solution set. Thus, equations may act like sentences in that two equations or sentences may have different surface structures but the same deep structure.

If two equations or two sentences have different surface structures, then, how can we know whether they have the same deep structure? The answer, according to Chomsky (who, incidentally, began his career as a mathematics major), lies in grammatical transformations: sets of rules which allow us to transform one sentence into another, keeping the deep structure invariant. Mathematicians would no doubt use the term 'isomorphism' (Greek *iso* = same, *morph* = structure). For example the sentence

Jack saw Mary

can be transformed into the passive voice (with the same deep structure) as

Mary was seen by Jack

by interchanging the subject and object, changing the form of the verb, and adding 'by'. This is the same sort of set of rules which transforms

$$3x + 4 = 7$$

into

$$3x = 3$$

and to

$$x = 1$$

Thus, in studying language by observing the rules of the 'transformational grammar', Chomsky is saying that language can be studied in much the same way as mathematics.

Chomsky is correct in pointing out that a sentence cannot be completely analyzed from observing the transformations alone, without an understanding of the deep structure. Mathematics

teachers have probably seen students 'solve' equations as in the following:

$$\sin y = x$$

$$y = x/\sin$$

Clearly these students have performed a certain transformational rule on the surface structure, but have failed to grasp the deep structure of the equation. This amounts to what many mathematics teachers label 'symbol-pushing'.

Chomsky goes further by asserting that some aspects of deep structure are innate, genetically transmitted, and common to all languages. It is therefore possible to apply transformations to a sentence in one language and to obtain a sentence in another language (i.e., to translate or paraphrase), leaving the deep structure invariant.

One of Chomsky's most controversial theories is that human beings have an innate language learning/inventing capacity, and therefore that all languages develop along the same lines. As rather trivial examples of this idea, we may cite the fact that the same sorts of grammatical categories, such as nouns and verbs, are found in all languages, or that all languages can refer to past or future time, form questions and assert positives and negatives. But even these simple universals have been challenged by some linguists and anthropologists.

The existence of such linguistic universals presupposes some underlying structure to human thought. Various experts have differing opinions as to the nature of these universals, but many of their descriptions include the foundations of mathematics and logic. If they are correct, the universality of certain logical principles lends support to the view of mathematics as a 'universal language'. If an algebraic equation such as

$$3x + 4y = 6$$

can be understood in both its surface structure and deep structure by speakers of any language of the world, then all people must have an inborn capacity to understand mathematics in the same way. It is in this sense that mathematics can be termed a universal language.

This view of the universality of logical thought has not gone unchallenged, however. We see in Chapter 7 how Benjamin Whorf

proposed that language is prior to logic, and that speakers of different languages may have very different mathematics.

Language learning and mathematics learning

To return to Chomsky's work, one fact often used as supportive evidence for his theory of the universality of language learning ability is the extraordinary talent of young children everywhere for acquiring language. Almost everyone agrees that children pick up language far better than adults, although the fact that language learning ability decreases with age has not been well explained. In fact, recent evidence has tended to qualify these views, and some anthropologists have found societies where they claim adults can learn languages as well as children do. Although the decline in language-learning ability may be well documented in speakers of most languages of the world, the existence of a single counter-example (as mathematicians are well aware) can destroy a beautiful universal theory.

The evidence of Piagetian studies supports the view that children around the world develop language skills in roughly the same way. This phenomenon is discussed by Fromkin and Rodman (1978); they state:

> Language is acquired by stages, and observations of children in different language areas of the world reveal that the stages are very similar, possibly universal. (p. 244)

But this same evidence is the first indication that language learning and mathematics learning are basically different.

> Before they can add 2 + 2 children are conjoining sentences, asking questions, selecting appropriate pronouns, negating sentences. Yet, children are not taught language as they are taught arithmetic. (p. 242)

The ability to learn mathematics, at least in Western cultures increases throughout childhood, while the ability to learn language actually declines. Developmental psychologists, especially the Piagetian school, are firm in their conviction that children must pass through several stages before reaching the level of formal thought necessary to master many mathematical

notions. But in the case of language acquistion, most Western children of age 1 are already proficient language learners.

In fact, learning to speak a language may possibly be best accomplished without the use of too much formal thought. Children do not need the analysis of language into nouns and verbs, nor the memorization of verb conjugations and noun declensions, in order to master the spoken language. On the other hand, we are all too familiar with adults who have studied a foreign language formally for several years but who can hardly carry on a conversation in that language, despite their ability to conjugate hundreds of verbs in every conceivable tense. Perhaps the time necessary for a formal analysis of tenses and endings is even a barrier to the spontaneous production of a sentence.

Understanding of a mathematical idea involves bringing it to a high level of consciousness - before one's eyes, as it were - in order to examine it from several points of view. Understanding of spoken language is done subconsciously; the words, grammar and structure need not be analyzed under the microscope in order to appreciate what is being said, although it must be admitted that literature is sometimes better understood by subjecting it to the scrutiny of literary criticism. But in general, the best way to learn a spoken language is usually through a total immersion into the flow of the language, while the best way to learn mathematics is through a detachment or distancing from the notions involved.

To go even further, we might even say that the understanding of a mathematical idea is an altogether different kind of meaning from the understanding of a spoken sentence. We shall be treating Halliday's (1978) discussion of "Learning how to mean" in more detail later, but for now it can be stated that 'meaning' has many dimensions, and that meaning in mathematics may be different from meaning in everyday language.

Thus, mathematics learning and language learning appear to be quite different, and in some ways even antithetical. This fact in no way contradicts Chomsky's view that the structure of both language and mathematics can be analyzed in the same way. It is quite possible to use the same methods of analysis on very dissimilar objects. It is necessary to distinguish between an object and a method of analyzing that object.

This method becomes rather difficult when applied to mathematics. Is mathematics an object in itself, or is it only a method for analyzing objects? In fact, it is both object and method at the same time. A homomorphism for example, is a method of analyzing the structure of a group, a way of comparing one group's

structure with another's. But a homomorphism can be considered as on object in its own right, and it is even possible to construct groups of homomorphisms, and then to analyze those groups using homomorphisms which map homomorphisms to homomorphisms, and so on.

Mathematics is concerned with structure, and mathematicians tend to feel that if two objects are isomorphic, that is, if they have the same structure, then they may be treated as identical. In the same way, mathematicians may feel that mathematics and language are similar in the sense that they have similar structures, as shown by Chomsky. But language involves much more than just structure. Consider the two statements:

Good day, sir
and G'day, mate.

These two greetings have the same grammatical structure, and basically the same meaning. But they are not identical - one is used in formal situations, the other in informal situations, particularly in Australia. A mastery of the English or Australian language must include the knowledge of which situations call for which kind of language.

Register

Language is a social phenomenon. The two greetings described above have a social 'meaning' quite distinct from their primary meaning as a greeting. The former conveys respect; the latter, intimacy. Linguists call this social context of language 'register'. Some linguists even restrict the term to meaning language suited to a particular occupation (e.g., Trudgill, 1983), but we retain the broader sense here. Occupational jargon is, however, an excellent example of speech applied to a specific environment. A teacher speaks to students in a different manner than to family members. Adults use different registers when talking to children than when talking to other adults.

The most common mark of a register is its unique vocabulary, a fact which should not be surprising, since certain social roles, especially occupational ones, must use highly technical vocabulary. What is more interesting is the way in which some registers appropriate words for their own use in

special ways, often as metaphors, for example, the stockbroker's phrase, "the gold price is heading south."

Vocabulary is not the only mark of register. Lawyers may use long sentences while mothers may use short ones. Novelists may use flowery language in their novels but terse, restrained prose in their business letters. The structure of a conversation, in terms of who speaks for how long, is also a mark of register, since a formal lecture will use a different register from an informal classroom dialogue.

The reason for this discussion of register, of course, is that the language of mathematics can be classified as a register. It has its own specialized vocabulary, and even uses many English words in non-standard ways. For example, a group in the mathematical sense is not the same as a group of corporations or a discussion group. Furthermore, the precise and restricted sentence structure of mathematics is not the same as the flowing language of literature.

For these reasons, it is becoming increasingly popular to refer to mathematics as a register, not a separate language. Of course, we may call it whatever we like without changing its nature, but the term register serves to focus on some of the problems of acquiring a technical jargon. It also leads away from the structuralist definitions of Chomsky and others, since registers reflect different uses of language. But most importantly, it raises one of the most basic difficulties of students, namely that they are often confused as to whether a teacher's statement is to be taken in the sense of its mathematical register or in the sense of everyday language. Pimm (1987) cites some good examples of this phenomenon, for example, the use of the word 'diagonal'. In the mathematical register it refers to a line segment joining two vertices of a polygonal, while in everyday speech it may refer to any oblique line which is neither vertical nor horizontal. A student may therefore state that a rectangle whose sides are vertical and horizontal has no diagonals (in the everyday sense), but if it is rotated by 45 degrees it has four diagonals.

The functionalist view

The use of different registers reflects the fact that language has many functions. The use of the formal register when saying, "Good day, sir" expresses more than a greeting; it uses language to express respect in a rather indirect way. Thus, a sentence may

have 'meanings', perhaps not even intended, other than the primary *cognitive* meaning of the thought being expressed.

Most modern linguists classify language uses into various categories. Halliday (1975) outlines seven types of language which children acquire :

1. Instrumental ('I want'): satisfying material needs
2. Regulatory ('do as I tell you'): controlling the behaviour of others
3. Interactional ('me and you'): getting along with other people
4. Personal ('here I come'): identifying and expressing the self
5. Heuristic ('tell me why'): exploring the world around and inside one
6. Imaginative ('let's pretend'): creating a world of one's own
7. Informative ('I've got something to tell you'): communicating new information

If we examine the functions of mathematics, or even of mathematical language, in light of this description, we see that mathematics usually performs only the last three of the seven functions. In this manner, mathematics acts like a language in some ways, but not others. Of course, any utterance may have more than one use at a time, and it may be very difficult to specify what uses are intended for an utterance. This is the problem of analyzing language in the classroom. The teacher may claim to be using mathematical language in its heuristic or informative function, but in practice his or her language may be excessively complex, with the aim of intimidating students (regulatory function) or demonstrating his or her superiority (personal function).

Even if we grant that mathematical language can be put to various uses, it is not so easy to vary the grammar of that language according to the use. True, the teacher may alter the complexity of the terms to some extent, but in general there are not many ways of stating a mathematical fact, due to the precision of mathematical vocabulary. A speaker would not speak to a friend about mathematics much differently from the way he would speak to the Queen of England. It is precisely this relative invariance across

contexts which I believe is a major difference between mathematics and everyday language.

Contextualization

The linguistic notion of 'context of situation' has been discussed in great detail, dating back to the well-known anthropologist and linguist, Malinowski (1923). Underlying this notion is the observation that language can only exist *only* in the context of a situation. Halliday (1978) treats this idea succinctly:

> We do not experience language in isolation, but always in relation to a scenario, some background of persons and actions from which the things are said to derive their meaning. This is referred to as the 'situation', so language is said to function in 'contexts of situation' and any account of language which fails to build in the situation as an essential ingredient is likely to be artificial and unrewarding. (pp. 28-29)

Mathematics attempts to decontextualize thought as far as possible. In its insistence on precise language and elimination of 'fuzzy thinking', it demands that a sentence should have no ambiguity. It is not desirable that a mathematical expression should have multiple meanings in various contexts. Terms like 'group', 'ring', and 'field' each have one precise definition and in order to avoid confusion, they should not be used as having any other meaning. In aiming for such precision, mathematical language is robbed of some of the rich, multi-layered texture of spoken language, which is often intentionally ambiguous.

The Zande language has been shown to be particularly ambiguous (Evans-Pritchard, 1963). For the Zande, it is desirable for every sentence to have a double meaning, or 'sanza'. Every remark has a rather innocuous surface meaning, but contains some cryptic message as well, which may even insult the listener:

> The great thing is to keep under cover and to keep an open line of retreat should the sufferer from your malice take offense and try to make trouble. The listener wonders whether he has really understood he

> cannot be sure, whether the words do have a nuance or someone imagines that they do. (p. 222)

Surely mathematical language and the Zande language are very different, at least functionally. It would probably be very difficult for speakers of Zande to learn mathematics in the Western tradition, since they are not used to confining their utterances to unique and precise meanings. Those same difficulties will exist to some extent for speakers of any language, because every language is used to express intentional ambiguities and hidden meanings.

The attempt by mathematics to isolate substance from context is what makes mathematical thinking difficult for many people. Everyday thought is inseparable from context. In some instances, a slight variation in environmental conditions may make the subjects of an experiment uncomfortable enough to be unable to perform tasks which the experimenter considers trivial and unconnected with that change in environment. This is why many researchers refuse to perform educational experiments in laboratory conditions. Students may be able to perform normally in the familiar setting of their classroom, but may be thrown by the new environment of the laboratory.

Actually, most people pay more attention to the affective message of language than to the actual text of what is said. A politician may be able to disguise a war-mongering hate campaign in the words of love and peace, with pictures of his loving family and pets in the background. Mathematical language, deprived of context, could not do this, and indeed strives to promote the ability to discriminate the content from the context.

There are linguists who claim that some groups of people are better than others at decontextualizing their thought. Chapter 10 describes the theory that lower-class children use much more context-tied language than do middle-class children, and are therefore at a possible disadvantage when it comes to mathematics learning. The important question is not just whether they are initially disadvantaged by their social background. More importantly, are they *permanently* disadvantaged in mathematics simply because of the language they use? Is it perhaps even *impossible* for lower-class children to learn a depersonalized mathematics in all its decontextualization if they continue to speak the context-related language which they have learned at home?

It is at this point that many modern linguists part company with the mathematicians. For they believe that content and context are inseparable. Here is Halliday (1978) again:

> We do not, in fact, first decide what we want to say, independently of the setting, and then dress it up in a garb that is appropriate to it in the context. The 'content' is part of the total planning that takes place. There is no clear line between 'what' and the 'how'; all language is language-in-use, in a context of situation, and all of it relates to the situation. (p. 33)

This description epitomizes the functionalist viewpoint. It shares some common ground with Marshall McLuhan's famous phrase: "The medium is the message." But is does not go so far as to say the content - the 'what' - does not exist. Both the 'what' and the 'how' can be said to coexist without being one and the same thing.

It is possible to draw an analogy with modern physics' description of light. Light consists of neither waves (medium) nor particles (content), nor can light be adequately described as particles riding on the backs of waves. It simply manifests some of the properties of particles and some of waves. Similarly, language is neither the content, nor the context, nor simply thought riding on the backs of words and phrases.

We have come full circle from the earlier discussion of mathematics as neither fully structure nor fully method. For now it is language which can be seen as neither structure nor method, in the sense that some aspects of language may be better viewed functionally while other aspects may be better viewed structurally. On the one hand, language is not simply a structure: a collection of words and phrases, together with certain rules which bind them. It goes far beyond that, in the sense that it is interactive and cannot be separated from its context. Tone of voice, emphasis, body language, and even personal appearance all contribute to the totality of the communicative process. This is why most linguists today refer not to speech but to 'speech events'.

On the other hand, language is not pure method: an utterance is not simply a vehicle for sending a pre-existing message. Theorists are distancing themselves from the simplistic notion of pure atoms of thought being transmitted from sender to receiver encoded in language. Language is somehow intimately bound up with the thought process itself. It is that 'somehow' that may be the key question in psycholinguistics.

The debate has persisted as to which comes first, thought or language. Chapter 7 discusses the extreme view that thought exists only in language and must therefore vary according to the language

spoken - Greek, English, or Hopi. This is the so-called *Whorf hypothesis*. The answer currently in fashion, however, is that neither language nor thought comes first: they come together, but the mechanics of how this is actually accomplished are still not well understood.

Thus, language and mathematics are similar in the sense that they are both neither pure structure nor pure method. The relationship between structure and method has been pursued by Inhelder and Piaget (1980), who realize that, indeed, structures and procedures are not isolated concepts:

> On the one hand, every mathematical construction is based on structures, structures which are currently used or yet to be invented. Using structures or inventing them implies the use of procedures which Polya (1945) termed "heuristics." On the other hand, any strategy used by a child to solve a practical intelligence problem implies the use of procedures that necessarily call on structural knowledge, either already acquired or discovered during the task. (p. 19)

Thus, every structure implies a procedure, and every procedure implies a structure. The two cannot be separated.

Mathematics teachers are language teachers

The result of this lengthy discussion is that mathematics and mathematical language cannot be separated. Mathematical concepts as objects and structures are not atoms of pure thought; they coexist with language in a very complex way.

I have heard mathematics teachers say that their job is to teach mathematics, not English. In other words, they are saying implicitly that mathematical thoughts and concepts exist independently from language. This is just a restatement of the view that language is simply a code for transmitting previously existing thoughts. The blurring of this distinction between thoughts and coding device has two consequences.

Firstly, mathematical concepts, while perhaps not being identical to language, do not exist in isolation from language. It is therefore impossible to teach mathematical concepts without teaching the linguistic context which surrounds them.

Secondly, teachers should move away from the notion of 'getting across' an idea, in the sense that the idea is a fixed object which can be transmitted, via language, to the students' minds. Such notions promote the definition of education as filling children's minds with knowledge in the same way as filling a bowl with water. If mathematical structures are inseparable from their use, students can only "know" mathematics if they can use it. We are very close to the famous quotation of Jerome Bruner (1960), that "education is a process, not a product."

Summary

It is impossible to distinguish between mathematics and mathematical language, for the two do not exist in isolation from each other. In this sense the answer to the question in this chapter's title is "yes, mathematics is language, or at least it cannot be separated from language." On the other hand, mathematical language can be considered as a register which is in many ways different from everyday language. The major difference is in the fact that everyday language, learned and used by children in the context of a situation, is in a natural way tied to the context of the situation, perhaps in emotional, 'irrational' ways, while the mathematical register tries to remove that link to the context as much as possible.

The suggestion that concepts and logic can be separated or abstracted from their context is indicative of the modern Western view; the view that they *should* be separated indicates a value system. Chapter 11 will discuss the mathematical/rational view as a value system. Mathematics teachers must be aware of the fact that they are teaching an ideology and that there may be other ideologies which are not compatible with the mathematical one.

Teachers in multicultural situations must also be aware that uses of language may vary from culture to culture. Teachers sent to teach abroad for the first time must be sensitive to cultural differences in language and in mathematics.

In short, language pervades our lives to a greater extent than most of us realize. The fact that almost all of us speak a language naturally makes us take our language for granted, so that we may be unaware of the many uses of the language we use. We now attempt to alert mathematics teachers and educators to some of the ways in which language affects mathematics education, both among first language and second language speakers.

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CHAPTER 2

THE DEVELOPMENT OF FORMAL THOUGHT THROUGH LANGUAGE: PIAGET VS. VYGOTSKY

This chapter studies how the child uses language to arrive at the stage of formal thought. Most theories view language as very strongly connected with formal thought processes, but there is disagreement as to which precedes the other. There are many theorists who have written on the topic, but the discussion here focuses on two authors who represent apparently opposing schools of thought. Both have produced well-written, cogent explanations of their theories, and both have backed up their theories with sound empirical evidence. Of particular importance to mathematics and science education are their views on the role of language development in the shaping of cause-and-effect relationships and the principles of logical inference.

Piaget, probably the most well known theorist on children's mental development, was a cognitive psychologist who found it necessary to study linguistics in order to understand cognitive development. It is perhaps natural, therefore, that he appears to put thought ahead of language: to view thought as existing prior to language, even though language is a vehicle for thought.

Perhaps less well known to educators is the Russian linguist L.S. Vygotsky, who did his seminal work in the early 1900s, but whose work gained wider recognition in the 1960s. In his study of language, Vygotsky found it necessary to study human cognitive development, and it is therefore not surprising that he views formal thought as a product of language development.

Finally, this chapter illustrates how some contemporary linguistic theorists attempt to synthesize the views of Piaget and Vygotsky.

Review of Piaget's developmental theory

Piaget's theories of development hold that children pass through a series of developmental stages leading to the final stage of formal thought. It is probably convenient, however, to review Piaget's theory briefly and to attempt to fix what is meant by the term 'formal thought'. As a discussion of Piaget's theory of language and thought follows, we must be constantly aware that his theory of development may greatly affect his theory of language. If we accept *in toto* his view of developmental stages, we must ask ourselves whether we must then accept his theory of language as well.

Piaget's theory of cognitive development features prominently in any textbook on developmental psychology (e.g., Liebert *et al.*, 1986) and is usually described as follows: the child passes through a series of stages by which he comes to relate himself to his world. These stages are:

1. The sensorimotor period (0 to 2 years)
2. The preoperational period (2 to 7 years)
3. The concrete operational period (7 to 11 years)
4. The formal operational period (11 through adulthood)

During the sensorimotor period, the child passes from reflex action, to imitation, to 'deferred imitation', to active experimentation, and finally to mental experimentation. This sequence illustrates the extent to which actions become internalized, in the sense that the child comes to visualize actions mentally without actually performing them physically.

Thought during the preoperational period shows an improvement in the ability to internalize actions. One of Piaget's major contributions to developmental psychology was to demonstrate that children are not just miniature adults, and that their logic is qualititatively different from that of adults. It is during the preoperational period that children display thought which has a consistent logic of its own. The most well-known example of this phenomenon is the irreversibility of actions as shown in the child's failure to 'conserve'. For example, a child is shown a beaker of water, and the water is then poured into a beaker of another shape. The child does not appreciate that the volume of water remains constant and that it could be poured back into the first beaker and appear exactly as it did at first.

Piaget's description of the preoperational period relies heavily on the notion of 'egocentric' thought. This term refers to

the child's inability to see the world from any perspective other than his own. The concept of egocentric thought is central to Piaget's theory of the development of language. As we shall see later, conversations between children at this stage are little more than monologues - relatively little actual communication takes place. Thus, early language in the child is more of an egocentric language than a communicative one.

The concrete operational stage sees the development of conservation and the mental transformation of objects. This includes the construction of sets and subsets within the mind, along with an understanding of the relationships between these classifications. But this type of thought, as internalized as it is, falls short of operating with abstract or hypothetical concepts - it can deal mentally with real objects, but cannot cope with "what if" situations.

It is this hypothetical thought with abstract concepts which distinguishes the formal operational period. An appreciation of deductive reasoning, of syllogistic logic, and of cause-and-effect logic appears at this stage. A classic example of Piaget's experiments is one in which a child is asked to experiment with a pendulum to discover what factors affect the length of time for each swing (the period). The child must be able to *hypothesize* factors which may or may not *cause* the pendulum to speed up or slow down. He must then *systematically test* his hypothesis in order to *deduce* that one factor (the length) and not some other factor is the deciding variable.

Weaknesses in Piaget's developmental theory

One of the good points of the above theory is that it does not exist in a vacuum: it can be tested empirically. However, subsequent experimentation has revealed several difficuties with the theory.

1. The theory appears to be a maturational one, in the sense that development through stages sounds like a biologically inescapable law. If so, then all children everywhere should pass through these stages at roughly the same age, or at least at the same stage of physical development. An excellent summary of cross-cultural Piagetian research can be found in Dasen (1972), who points out that while the existence of the various stages appears to be nearly universal, their timing is not. The age of attainment of the concrete operational stage, for example, varies

considerably from culture to culture. Indeed, according to Dasen "some children, and even adults, do not reach the concrete operational stage."

Thus, development through the stages appears to be strongly affected by environmental and cultural conditions. THis experimental finding belies the biological inevitability of the Piagetian stages.

Piaget answered this criticism by stating that while his theory is essentially maturational, the attainment of the various stages can be delayed, or possibly even accelerated by environmental conditions. Indeed, research has faithfully documented that the *order* of the developmental stages remains constant in different cultures, even though the actual *timing* varies, just as all 'normal' children learn to speak along certain lines, although some children learn to speak much earlier than others due to environmental factors.

2. If a child is at a certain stage at a given moment, say the concrete operational stage at age nine, one would expect him to perform consistently at that level over many tasks. For example, he should display conservation of liquid in the beaker experiment, as well as conservation of mass in a weighing experiment. Research has shown, however, that children may perform at one developmental level in one set of circumstances, and at another level under different circumstances.

Piaget responds that his theory is in no way damaged by such observations. The child may pass through the various stages at differing rates in different domains. A child may display preoperational behaviour in playing with toy cars, for example, but concrete operational behaviour in dealing with, say, animals. Piaget calls this varying behaviour 'decalage'. Decalage is simply an extension of his argument that environmental factors can speed up or retard the developmental process. For if a city child, for example, grows up in an environment heavily influenced by machines and automobiles, he may learn to deal with these objects in a concrete operational way at a relatively early age.

We must realize that resorting to the notion of decalage, while saving Piaget's overall theory, weakens it substantially as a theory. For the strength of a theory lies in its ability to predict and generalize. The notion of decalage greatly weakens the predictive value of Piaget's developmental theory, since we can no longer be confident of a child's behaviour on a task. The first hedging of bets - the acceptance that environmental factors affect

the rate of development - makes it impossible to predict a child's behaviour simply on the basis of age or physical development. The second hedge - that of decalage - states that even if we observe a child behaving at one stage on one task, we cannot predict his behaviour on another task.

In summary, then, Piaget's theories of developmental stages have been experimentally shown to be essentially correct, but because of the differences in timing, and because of the decalage factor, the theory is diluted so as to be of little predictive value. The addition of cultural and other variables into the picture renders the question of child development much more complex than the pure Piagetian maturational theory seems to indicate.

Descriptions of language

Before we begin comparing Piaget's theories of language development with those of Vygotsky, it is useful to point out several areas of agreement, not only between these two theorists, but among virtually all theorists. For example, it is easily verifiable that much of young children's language is egocentric, and that the percentage of egocentric language decreases as the child grows to adulthood. We shall return to the divergent explanations for this egocentrism in our comparison of the two theorists. Underlying all theories of language development must exist some notion of the functions of language. On this point there is considerable agreement, at least on the most general ideas.

Almost all experts on language development in children distinguish between two major functions of language - an internal function and an external one. Each linguist seems to adopt his or her unique name for these two functions, but the descriptions remain roughly the same. Piaget (1926) describes them as follows:

> At times, language serves only to assert, words state objective facts, they convey information, and are closely bound up with cognition. "The weather is changing for the worse," "Bodies fall to the ground." At times, on the other hand, language expresses commands or desires, and serves to criticize or threaten, in a word to arouse feelings and provoke action - "Let's go", "How horrible!" etc.

Closely connected with these two functions of speech are Piaget's notions of *socialized speech* versus *egocentric speech.* This distinction is central to his theory of language,and so we must examine it in detail in order to comprehend fully the remainder of the theory. Piaget himself spends considerable effort on explaining egocentric speech, so we quote him directly:

> When a child utters phrases [of egocentric speech], he does not bother to know to whom he is speaking nor whether he is being listened to. This talk is egocentric, partly because the child speaks only about himself, but chiefly because he does not attempt to place himself at the point of view of his hearer. He has no desire to influence his hearer nor to tell him anything.

By contrast,

> In socialized speech we can distinguish adapted information: Here the child really exchanges his thoughts with others, either by telling his hearer something that will interest him and influence his actions, or by an actual interchange of ideas by argument or even by collaboration in pursuit of a common aim.

Note that the above distinction has not yet become a theory. It simply serves as a context for discussion. There is almost nothing with which we can disagree. Indeed, Vygotsky accepts the framework and replicates some of the same experiments as Piaget as to what fraction of children's speech belongs to one category or another. Both Piaget and Vygotsky arrive at similar figures which have been verified independently in other studies. It appears that around the age of 6, the ratio of egocentric speech to socialized speech is around 0.45. By the age of 7 or 8, this ratio drops rapidly to around 0.27. This change in ratio represents empirical findings which must be explained by a true theory, and it is at this point of explanation that Piaget and Vygotsky diverge.

Piaget sees egocentric speech as a reflection of egocentric thought, which is a bridge between *autistic thought* and *communicated intelligence.* Autistic thought is:

> ...subconscious, which means that the aims it pursues and the problems it tries to solve are not present in consciousness; it is not adapted to reality, but creates

> for itself a dream world of imagination; it tends, not to establish truths, but so to satisfy desires, and it remains strictly individual and incommunicable as such by means of language. On the contrary, it works chiefly by images, and in order to express itself, has recourse to indirect methods, evoking by means of symbols and myths the feeling by which it is led.

This succinctly expresses much of Piaget's theory of language. It states that thought exists prior to, and outside of, language.

The preoperational child engages largely in autistic thought, but he needs to adapt to the external world through language. Egocentric thought is the first step in internalizing the outside world, and egocentric language is a tool for doing so. When this process of adaptation is largely complete, the need for the tool of egocentric language diminishes, and so the fraction of speech devoted to egocentric speech drops rapidly around the age of 7 or 8.

As noted above, Piaget is well known for his view that children are not simply miniature adults - that the logic of the child is *qualititatively* different from that of adults. This difference is primarliy the difference between egocentric thought and 'intelligence'. Egocentric thought retains some of the characteristics of autistic thought but goes some way towards intelligent thought. It is not as though a child uses autistic thought one day and egocentric thought the next. The change is gradual, and the boundaries between the classifications are not clear-cut, but certain tendencies can be described.

> Egocentric logic is more intuitive, more 'syncretistic' than deductive, i.e., its reasoning is not made explicit. The mind leaps from premise to conclusion at a single bound, without stopping on the way. Little value is attached to proving, or even checking propositions.
>
> In communicated intelligence, on the other hand, we find far more deduction, more of an attempt to render explicit the relations between propositions by such expressions as *therefore, if....then*, etc. Greater emphasis is laid on proof. Indeed, the whole exposition is formed in view of the proof, i.e. in view of the necessity of convincing someone else, and of convincing oneself.

Piaget goes on to claim that egocentric language and logic have no use after a certain age, and thus die away. But we are all familiar with the adult phenomenon of 'bouncing ideas off people'. When we have a problem, logical or emotional, we benefit by talking it over with someone else. This is because, as Piaget states above, logic is carried out in the framework of convincing ourselves just as though we were convincing someone else. This kind of *adult* speech is very much Piaget's egocentric speech, as it is performed for the purpose of organizing our own thoughts.

Would it hurt Piaget's theory very much if he were simply to admit that egocentric speech dwindles but never quite disappears? I believe it would. For to the extent that egocentric speech is an indicator of egocentric thought, such an admission would indicate that adults still use egocentric thought from time to time. In fact, the proportion of adult speech which is egocentric in nature can be measured empirically. To my knowledge, this has never been done., but I suspect that it is higher than Piaget would like to believe. Statements like "I think I'll go to town now" may in many circumstances not be intended for the listener, but may be designed to organize our course of action, and should therefore be classified as primarily egocentric.

But the more damaging criticism would be if one were to show that the more formal the logical setting, the more the adult resorts to egocentric speech. For that would indicate that egocentric speech is an indicator not of egocentric thought, but of intelligent thought. Think of solving a mathematics problem. If the problem is a simple one, the solution may present itself immediately. But if the logic is complex, we often restate the problem to ourselves and actually talk over the problem in our minds. Is this not egocentric thought? We shall return to this criticism when we discuss Vygotsky's views.

While we are discussing Piaget's notion of egocentric versus intelligent thought, we need to examine in more detail his above statement concerning the use of logical connectives such as 'if-then'. When and how the child uses these connectives is of utmost importance to their learning of mathematics and science, and so Piaget's views here should be the most important ones for teachers and theorists.

Reason and consequence

When considering connectives such as 'why', 'because', 'if-then', we must distinguish between *reasons* and *causes*. For example, suppose a child asks "Why did the stone sink to the bottom of the lake?". We could give two possible answers:

1. "Because I threw it in"; or
2. "Because dense objects sink".

The first answer is a cause: the rock's sinking followed and was a result of my throwing it in the lake. The second answer is an explanation or a law.

Piaget's experiments showed that these two kinds of reasoning are quite different in children, and do not necessarily develop at the same stage. In science and mathematics education, it is the 'reason-consequence' type of 'because' which is of interest, and Piaget showed that children under the age of 7 rarely use the word 'because' in this way. Other researchers, including Vygotsky, have confirmed this finding. We must therefore ask why such reasoning appears so late in a child's speech.

Piaget himself realizes that this question is of fundamental importance. In particular, does the notion of reason and consequence exist in a child's mind at all, and if so, why is it manifested so late? Piaget is clear in his answer: the concept exists but the egocentric child has no desire to express it:

> Does the absence of verbal forms expressing logical relations prevent genuine argument from manifesting itself, or does the absence of the desire to argue and collaborate explain the late appearance of these verbal forms? If we admit that thought in the child depends upon his interests and activities rather than *vice versa*, then the absence of the desire to argue and collaborate is obviously the initial factor.

This answer is entirely consistent with Piaget's theories expressed earlier. The egocentric child has no desire to convince anyone else of a point of view, and therefore he has no occasion to explain general rules using the word 'because'. He may only use the word in the sense of "because I threw it in", i.e. when one event follows another. Similarly, since the child rarely seeks information from others, he will only rarely ask "Why?"

Piaget offers clear evidence in support of his answer by showing that children *do* use logical connectives in conversations

with adults. This finding is in line with the view stated above that thought is prior to language: it is only when the child becomes interested in expressing the thought that he adopts a certain language. This point seems so self-evident that it may be surprising that Vygotsky, as we shall see, disagrees at precisely this juncture. Bearing this disagreement in mind, we now turn to the theories of Vygotsky.

Linguistic theories of Vygotsky

Vygotsky (1962) accepts Piaget's data and even replicates his experiments, obtaining similar results, but he interprets them in quite the opposite way. Where Piaget sees the development of thought and language as proceeding from autistic to social, Vygotsky sees the progression in the opposite direction. He claims that language in its first stages is strictly a social phenomenon: the infant communicates its desires to its mother and learns about the external world from her. Young children have no concept of self as a distinct entity. It is with the aid of language that they develop such a concept and adapt that self-concept to their social environment. Vygotsky could not be more explicit on this point:

> The primary function of speech, in both children and adults, is communication, social contact. The earliest speech of the child is therefore essentially social. Thus our schema of development - first social, then egocentric, then inner speech - contrasts with Piaget's sequence: from nonverbal autistic thought throught egocentric thought and speech to socialized speech and logical thinking. In our conception, the true direction of the development of thinking is not from the individual to the socialized, but from the social to the individual.

It is remarkable that such opposing theories could be based on virtually the same empirical evidence. The primary reason for this opposition is a difference in the fundamental view of the function of language. For Piaget, the thought exists prior to language and is simply socialized using language. For Vygotsky, on the other hand, language shapes the thought. For example, we have seen that Piaget argues that the logic of consequences exists in the child's mind, but that the child has no desire to express it.

Vygotsky, in contrast, claims that this logic develops *through* language. The child, slowly becoming aware of his own existence, internalizes experience using language.

We must be careful not to interpret this view as claiming that thought is synonymous with language, or that thought *is* language. Rather, language is a tool which renders thought easier, in the same way that a mathematical equation such as 2 + 3 = 5 is merely a tool for simplying a complex thought process.

In support of his theory as opposed to Piaget's, Vygotsky cites evidence that in situations where children's free thought was interrupted by small frustrations, the level of egocentric speech increased enormously. The interpretation of this observation is that egocentric speech is used to internalize and make sense of the outside world. For example, a child who was not given a blue pencil, said to himself, "Where's the pencil? I need a blue pencil. Never mind, I'll draw with the red one and wet it with water; it will become dark and look like blue." This is clearly language used as a tool to focus thought towards a solution to a problem, although such utterances may be argued to have a certain 'display' function as well.

But if egocentric language is a tool for thought, should we not expect it to increase into adulthood, as thought becomes more complex? How does Vygotsky explain the disappearance of egocentric language as observed by Piaget? The answer is quite simple: egocentric language does not disappear - it 'goes underground' and becomes 'inner speech'. Thus, even adults use language to think, and so in some way they talk to themselves. They have just become so adept at it that they seldom need to express their thoughts aloud. This argument goes a long way towards explaining why adults occasionally use egocentric speech in situations where a problem is logically complex and must be 'talked out'. The view is entirely consistent with the direction of development from social to personal thought, or as Vygotsky puts it:

> The inner speech of the adult represents his 'thinking for himself' rather than social adaptation; i.e., it has the same structural characteristics: out of context it would be incomprehensible to others because it omits to 'mention' what is obvious to the 'speaker'. (p. 18)

Another empirical finding of Piaget's which Vygotsky must answer is the example, already cited, of children's ability to use logical connectives when talking with adults, but not desiring to do so when talking with other children. For does this finding not show that logical thought is already present before it is manifested in egocentric language?

Vygotsky's response is actually to deny the evidence. He claims that children do not understand the logical connectives they are using, but instead are mindlessly imitating adult speech because they believe that is what adults want to hear from them. This argument sounds as though he is avoiding the issue, but in fact he does provide empirical evidence to support his contention. He performs experiments which indicate that young children do not understand the difference between reason and consequence: between 'if' and 'then'. They often make errors in the order of propositions, for example, "I don't like my breakfast because I won't eat it."

Support from neurophysiology - A. R. Luria

Vygotsky's theories are supported by the prolific work of the other giant of Russian psycholinguistics, A. R. Luria (1966) . Luria was essentially a neurophysiologist who attempted to theorize about higher mental processes and language, based on physiological research into the human brain. Like Vygotsky, Luria was impressed by the importance of language in the formation of cognitive structures:

> Speech plays a decisive role in the mediation of mental processes....The reorganization of mental activity by means of speech and the incorporation of the system of speech connections into a large number of processes, are among the more important factors in the formation of the higher mental functions, whereby man, as distinct from animals, acquires consciousness and volition. (pp. 33-34)

One of the principal methods of the neurophysiologist for studying the function of the mind is the study of pathological cases: instances of lesions in one part or another of the brain. If a lesion in one area leads to a certain deficiency in behaviour, it might be concluded that the center for that particular behaviour is located at

the site in the brain where the lesion has occurred. While this method sometimes is useful, it often leads to paradoxical results, for it is based on an assumption that thought is localized.

In fact, Luria was led to support the developmental theories of Vygotsky because of observations that as children with certain brain lesions matured, the effects of the lesion seemed to change.

> The structural variation of the higher mental functions at different stages of development means that their cortical organization likewise does not remain unchanged and that at different stages of development they are carried out by different constellations of cortical zones. The character of the cortical intercentral relationships does not remain the same at different stages of development of a function and the effect of a lesion of a particular part of the brain will differ at different stages of functional development. (pp. 36-37.)

An important corollary of this theory is that thought is not localized in any particular site in the brain. Note the use of the phrase 'constellations of cortical zones' in the above. It implies that thought occurs as a combined effort from several areas in the brain. Mathematics educators may be fond of talking of 'right-brained' or 'left-brained' activity, but Luria's theory warns against such talk. While certain mathematical activity may properly said to be predominantly left-brained, that activity may arise from several areas of the brain, perhaps sometimes in the right hemisphere.

> If the higher mental functions are complex, organized functional systems that are social in origin, any attempt to localize them in special, circumscribed areas ('centers') is [not] justifiable. We therefore suggest that the material basis of the higher nervous processes is the brain as a whole, but that the brain is a highly differentiated system whose parts are responsible for different aspects of the unified whole. (p. 35)

Note that Luria agrees with Vygotsky in assigning a social origin to higher mental functions. Both theorists agree that the language of society comes first, that this language is transmitted to

the child, and that the child can then use the language to shape the higher mental functions. This is in opposition to the Piagetian view that the thought exists first and that the child learns the language of society in order to transmit those thoughts to others.

In treating language and thought as social phenomena, Luria and Vygotsky anticipated the sociolinguistic school of thought which is in favour today.

Synthesis: the contemporary sociolinguistic approach

The theories of Piaget and Vygotsky seem diametrically opposed. One states that thought precedes language, while the other claims that language precedes thought. How, then, can a synthesis possibly be created which incorporates both views? The development of meaningful thought must proceed either from the personal to the social, or from the social to the personal, but certainly not both ways.

Modern linguistics has moved in the direction of viewing not only all language, but all *meaning* as social. In so doing, it obviates the need for discussing the Piaget-Vygotsky question altogether. Such notions as concepts and logical operations are no longer considered to be attributes of the individual; they are social phenomena which belong to a society. The learning of the child is simply the acceptance of society's conventions. For a more comprehensive discussion of these theories, let us now turn to the important work of M.A.K. Halliday (1975).

Halliday's primary interest is a linguistic one: the acquisition of language. We might therefore expect a certain bias in the direction of explaining all phenomena as based on language. But as a sociolinguist, his emphasis is often on the 'socio', and he sees language as simply one form of the communication of meaning in society. Modern sociolinguists do not treat language as a set of rules, words, and phrases. Rather, they treat it in the context of the entire gamut of devices used by humans to convey meaning. The study of the communication of meaning is the science of semiotics; hence Halliday can talk of language as a 'social semiotic'. But this term is simply a word which indicates a perspective from which to study children's language development; it is not a theory as such.

As in the discussion of Piaget and Vygotsky, it is useful to look first for grounds of agreement between them and Halliday. Halliday, too, is an empiricist, and, like Piaget, who collected much of his data from observations of his own children's

development, bases his theories on the detailed observations of his son, whom he calls Nigel in his writings.

Halliday is in full agreement with the notion of two types of language in children. He calls them 'pragmatic' and 'mathetic':

> One of them [pragmatic] demands a response, the other not. The response that is is demanded is, at first, in terms of goods and services: 'I want that', 'Do that again' and so on. Increasingly it becomes a demand for a verbal response, e.g. 'What's that?' The other type of utterance [mathetic], which demands no response, involves at first the observation, recall or prediction of phenomena seen or heard: 'I can see/hear'; it then extends to narrative and descriptive contexts.

These two functions are virtually identical to Piaget's egocentric speech (mathetic) and social speech (pragmatic). The two functions are so clearly delineated that Halliday claims his son Nigel always used a rising tone when using the pragmatic mode, and a falling tone when using the mathetic mode. However, it must be made clear that mathetic speech is in no way identified with egocentric thought. Moreover, Halliday is describing Nigel at the age of 19 months, while Piaget's descriptions are mostly of children of ages 6 to 7 years.

In fact Halliday believes that after the second year of a child's life, the mathetic-pragmatic dichotomy dies away, or rather it is replaced by a new function which is at once pragmatic *and* mathetic. It is when the child learns to combine the two types of language that he develops 'meaning'. From this stage onwards, there are no purely pragmatic or purely mathetic statements - every utterance has an element of both.

This notion of 'meaning' is not a new one. Indeed, one of the pioneers in the theory of language development in the 1920s, William Stern (1928), expressed roughly the same idea. He stated that at the age of around 2, the child "acquires the ability to mean something when uttering sounds, to refer to something objective."

Words such as 'meaning' and 'objective' are difficult to define. They both have something to do with the awareness of the individual of his or her position as simultaneous actor and observer. Stern used the term 'intentionality' to describe the phenomenon.

Halliday discusses 'meaning' as related to the context of a situation. A robot can be programmed to emit words which give a given message, but does not 'intend' as a conscious act. The human, in constantly observing the context of the situation has a large range of options by which he communicates meaning, including gestures, facial expressions and dress. Language, in Halliday's view, is not simply a message which can be handed over in one piece to the listener; it is a constant flow of assessments of the context with concomitant alterations and changes of strategy as the context changes. If the listener, for example, frowns during a sentence, the speaker may change his mode of speech in mid-sentence. Similarly, a given speaker will use vastly different language when communicating with different audiences, and must constantly reassess the reactions of the audience - whether he is 'getting his idea across'.

Halliday uses the term 'text', and claims that adult speech consists of not two but three functions: the ideational (cf. Vygotsky's 'inner speech'), the interpersonal (cf. 'socialized), *and* the textual:

> In the first place, text is meaning. It is not made of sounds or letters. It is made of meanings, and encoded in wordings, soundings, and spellings. Secondly, text is choice. A text represents a selection within numerous sets of options; everything that is said presupposes a background of what might have been said but what was not. The microlinguistic acts, or countless small choices that the speaker makes as he goes along, are actually microsemantic acts. (p. 123-124)

In this way, a sentence on its own has no meaning. Meaning can only exist in a social context. Even a simple conceptual word like 'red' has a social meaning. Sociolinguists no longer say that an individual possesses the concept 'red', for the concept only exists when applied in communication. In one context, my 'red' may be different from the 'red' which I use in another context.

In general, the term 'red' belongs to its culture, and language is of course inextricably bound up with culture. When I use the term, I have a fairly good idea of how it will be understood by other members of my culture. In this context, language learning is therefore simply learning how other people respond to

linguistic and other signals. The concept 'red' therefore belongs to the culture and not to the individual.

Note that these ideas apparently resolve the dispute between Piaget and Vygotsky over the child's use of logical connectives. For the child learns to use them in the context of speaking with adults, but has no need for using them with other children. In this sense Piaget is correct. But Vygotsky is also correct in his observation that children at first often use the connectives 'incorrectly'. In learning to use language, the child must make errors at first, or to put it in the language of Halliday, the child gradually refines his knowledge of situational contexts so that he can manipulate the language more effectively. In this sense we cannot say that a given use of the word 'red' is actually incorrect, since there is no one single red which applies to everyone. We can only say that a use of the term more or less effectively communicates what members of the culture generally expect it to mean.

Note also that Halliday, in claiming that language is essentially social, is partially siding with Vygotsky. His views must admit at least the possibility of linguistic relativity, for example, that laws of logic, of consequence, of cause and effect are specific to a given culture, and that when I say "A implies B" to a member of another culture, I may be communicating a concept which does not coincide very well with what his culture understands by the phrase.

How would Halliday interpret egocentric speech? That certainly is not communicating any social semiotic. His answer would probably be that the child's egocentric speech is in some sense, communicating with himself. Awareness of self entails the ability to stand outside oneself and to treat the self as another person. This is what consciousness is all about. Conscious thought involves a dialogue with one's self, or as Janet and Rignano (cited in Piaget, 1926) have stated it:

> All reflexion is the outcome of an internal debate in which a conclusion is reached, just as though the individual reproduced towards himself an attitude which he had previously adopted towards others. (p. 75)

Although I basically agree with Halliday's assessment, I believe he overemphasizes language as communication between people and does not go far enough in his explanations of what happens *within* an individual. It is all very well to say that

language is a social semiotic when used between individuals, but it is also true that an individual talks with himself. Halliday does not discuss inner speech adequately, in my view, and leaves unanswered some rather obvious questions. For example, when I solve a problem for myself, using, say, the concept of 'red', I must have some idea of what I mean by the term, and thus, in a very real sense, the concept of 'red' belongs to me. Is my personal concept of 'red' the same as that used when I speak with others? Similarly, when I use written language to address an unknown audience, what does my word 'red' actually 'mean'?

Wells' description of written language

One account of written language which appears to extend Halliday's discussion is that of Wells (1981). His description of the development of verbal speech is almost identical to that of Halliday's, but he goes further in describing the personal aspect:

> While the first two phases of development are characterised by the close interrelatedness of language, action and social context, the third phase sees a loosening of these links, as the child becomes conscious of his own mental states and capable of reflecting on his experience. As thought and action become separable, the function that language performs of representing the objects and events of experience is drawn upon to provide a 'tool for thinking' and a means for communicating to others the results of the thinking process.

Actually, written language is merely an extension of the formal abstraction process as described above. It functions as a 'tool for thinking'. We come back to the issue of literacy's effect on cognitive ability when we discuss the 'alphabet effect' in a later chapter. But it is probably fair to say that Wells and other sociolinguists view written language as not very different from socialized speech acts. This view is reflected in results of research by Athey (1971):

> The overriding principle or agreement may be summarized in the statement that reading is not a "skill" or even a bundle of skills, but a system of social

> communication. This statement...means that reading is viewed rather as one part of the fabric of living and problem solving....It involves the realization that the printed word is another system of communication analogous to speech.

Thus the writer as communicator is still aware of who his audience is, although to a lesser degree than in the case of speech, and so writing is still a tool for communication. It still belongs to the category of social semiotic. Although writing may differ from oral speech in degree only, that degree may be so great that it requires considerable effort for the child to go from oral speech, where communication in a concrete situation to a visible audience, to a decontextualized written speech where the audience is itself an abstraction.

Pedagogical consequences

The theories discussed here pertain mostly to young children. The evidence points to language and thought before the age of 7 as egocentric. Teachers must therefore be aware that what appears to be dialogue may be simply 'collective monologue' (Piaget's term). For a teacher to simply to talk to a child may not penetrate; it is probably better to allow structured playtime in which the child may experiment with objects and concepts and talk to himself about what is happening. Vygotsky's observation that egocentric talk increases when frustrations or problems arise indicates that problem solving activity is important for a child's cognitive and linguistic development.

If we accept Halliday's view that formal language is a step beyond pragmatic and mathetic speech, it makes sense for teachers to lead children to unite these two kinds of speech. For example, teachers should consciously mix the two types of speech, e.g. "Tell me how to tie your shoe". The 'tell me' is pragmatic, while the description of tying the shoe is mathetic, and the child must then communicate his thoughts to the teacher. The emphasis, of course, is on language as a two-way street; the child must be an active participant in the language process.

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CHAPTER 3

CONCEPTS

Chapter 1 discussed ways in which mathematics is language. In those respects, the teaching of mathematical concepts is language teaching. In particular, concepts can be named, and so in many ways concept learning is related to vocabulary building. Research has been carried out on how the naming of concepts affects the learning of those concepts. This chapter discusses such research, but also explores the question of how the structural hierarchy of concepts affects cognitive development.

Words and concepts

One of the primary questions raised by the Piaget-Vygotsky debate is the nature of concepts. Does the concept exist prior to language, or does the label define the concept? Could we possibly develop the notion of a dog, for example, if we did not have a name for it?

This question has been debated long and hard, but the experimental evidence now appears to point decisively toward the view that young children can form concepts even before they learn to name them. The current view has been stated strongly by Lenneberg (1967):

> The abstractness underlying meanings in general may best be understood by considering concept-formation the primary cognitive process, and naming (as well as acquiring a name) the secondary cognitive process.

Work by Nelson (1973) and by Ricciuti (1965) produced strong evidence that very young children (from 12 to 24 months) could group blocks according to size and shape without using labels of any kind. This finding is evidence that the concepts of size and

shape can exist in the mind without the accompanying linguistic tag. Even animals can be taught concepts; well-known experiments with chimpanzees have shown that concept formation is indeed possible. In one such example (Premack and Premack, 1972), a chimp named Sarah was able not only to recognize symbols for objects, such as a small red square as a symbol for 'banana', but also to form the concepts of colours. Sarah was able to use a small grey chip to represent the colour red, for instance.

It must be accepted that not all concepts are attached to language. But that acceptance is a far cry from stating that in every case the concept precedes the label.

That concepts can exist without language is quite reasonable. Deaf mutes get by in daily life with a minimum of difficulties. They recognize common objects such as chairs and doors without giving them names. Even when presented with a chair of a shape they have never seen before, they can figure out that the object is something to sit on, and thus they have learned the concept of 'something to sit on'.

In fact, if we accept Lenneberg's view that concept formation in the sense of recognizing similarities is the primary cognitive process, then the learning of words may be considered just one of the many kinds of similarities. When a child observes that other people use a given word in association with certain objects, the child simply classifies those objects as similar, since they all have the same name in common. The child may fail to see any other similarities, but the word alone suffices to link the objects as similar.

Another view of concept formation is that of the 'prototype' concept. In this view, the mind forms a perhaps vague image as a prototype or best example of a certain class. The concept is therefore not a set of criteria against which an external object can be judged as either belonging or not belonging to the class. Rather, the object is judged according to how much it resembles the prototype. Some objects resemble the prototype more than others. These fuzzy prototypes cannot easily be broken down into components for analysis, according to Posner (1973), but more recent work by Rosch and Mervis (1975) seems to indicate that even young children can analyze their prototypical referents. Bowerman (1975) argues:

> The sequence in which complexive categories structured around a prototype appear to develop as follows: The child hears a word modelled most frequently (often

> exclusively) in connection with one referrent or a small group of highly similar referents: e.g., 'moon' as the child looks at the moon or at pictures of the moon. The child's first use of the word also occurs in connection with one of these referents. After a variable period of time, the child begins to extend the word to referents that are similar to the original referent(s) in specific and consistent ways. (p. 251)

In this way, the child has a vague picture of a moon in mind, but it is impossible to state whether that picture is crescent-shaped or not. It is somehow a composite picture of all the moons the child has seen, condensed into one.

We may ask, of course, whether the child, who sees the moon as crescent-shaped one night and perfectly round another night, understands that it is the same object he sees night after night, or whether the collection of slightly different moons is seen as a set of discrete objects. But the idea of the moon as prototype is still a reasonable one, in that the child may see a picture of a crescent or a circle and call it a moon.

Varieties of concepts

The implication of the above arguments for teachers is that mathematical concepts may be learned in different ways. Children may form their own concepts by observing similarities or patterns. On the other hand, they may learn ready-made concepts through the medium of language. In the latter case, the name itself may be the sole unifying element of the concept, and the child may not be able to express any other reason for classifying the objects in a single category. This is why many teachers prefer the 'discovery' method, which permits students to form concepts on the basis of their (the students') own perceived similarities. In this case the word is secondary and serves to solidify the concept already learned. The items belonging to the concept are linked in the child's mind by a double bond: both a perceptual one and a linguistic one.

Now if concepts can be learned in different ways, the question arises as to which way is the most effective. Do students learn more effectively from perceiving similarities, or from learning ready-made labels? Evidence indicates that the most

effective method of concept learning varies from concept to concept and from task to task.

Ranken (1963) considered the task of putting together a jigsaw puzzle. In searching for a piece, is it better to form a mental image of the piece, or should one describe the piece verbally, such as , "The knob at the end looks like a duck's beak"? Ranken found that mental imagery was superior to verbalization, and in fact that forced verbalization could even hinder performance.

On the other hand, if a verbal stimulus is given, then verbalization can be effective in solving similar problems. Finding a piece in a jigsaw puzzle requires only short-term memory and relatively small distances in the search. In a widely cited article, McNeill (1965) postulated that linguistic encoding has its primary effect on memory, but that a concept consists simultaneously of a visual image and a linguistic label. Thus the word 'triangle' conjures up the mental image of a triangle, and a picture of a triangle also brings the word 'triangle' to mind.

Since long-term and short-term memory have been shown to have different properties, the long-term effect may be different from the short-term one. In remembering the piece of the jigsaw puzzle after one week, the verbalization 'duck's beak' may remain long after the actual shape has been forgotten, and in this case, the verbalization may be the most effective method of recalling finding the desired piece. Or perhaps the original precision of the mental image has been transformed to look much more like a duck's beak than it did one week earlier. Thus the use of the linguistic tag may have sacrificed precision for longevity.

Similarly, a language may be useful if the items in the class are spatially separated. Brown and Lenneberg (1954) asked children to pick out similar colours in a large array. They found that if children could name the colours they were searching for, they were usually more succesful in finding the correct match. Thus, the linguistic tag aided children in their search procedure, especially if the two coloured squares which matched were far apart.

To summarize these results, it can be said that there are many different kinds of concepts. Language may be central to some of them, peripheral to others, but most concepts are best learned if a combination of language and other factors such as visual imagery is present. Language is especially important for recognizing patterns which are separated by distance or by time.

Mathematical concepts

It appears that mathematical concepts are often quite distinct from those which occur in daily life. These two categories differ in many fundamental ways:

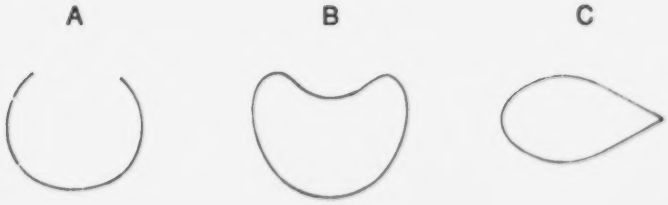
First, mathematical concepts are usually well-defined by a fixed definition, while everyday concepts can be quite 'fuzzy' indeed. Consider the everyday word 'round'. All children will claim they know the meaning of the word, and in most cases thay will be able to distinguish 'round' objects from other objects. But there are some plane shapes which may cause them some difficulty, for example the shapes shown below.

Some children may state that both are round. But others may claim that shape A has a pointed edge and therefore is not round. Others may claim that shape B is not round, but will be unable to put into words why it is not. Almost all children will state that some objects are more round than others. Thus the everyday concept 'round' is fuzzy in the sense that it not immediately clear whether some objects are round or not. Even if one is sure that an object is or is not round, that person is unlikely to be able to state precisely why. Can we claim that a person 'knows' a concept if he can identify examples of it without being able to explain why?

Mathematical concepts, on the other hand, are guided by definitions, so that it is possible to state precisely whether or not an object belongs to a class. The boundaries are by no means fuzzy. What is more, one of the objectives of mathematics teaching is to bring students to articulate exactly why an object does or does not belong to a given class. One could expect to see in a mathematical textbook a definition of the form:

> round: of, or pertaining to, a closed, convex, continuously differentiable plane curve, no subset of which is a straight line.

The usual pedagogical method is to state the definition and to follow it with examples which illustrate its finer aspects. Negative instances - examples which do not satisfy the definition - are usually useful. The objective is naturally to clarify the boundary of the defined category as precisely as possible in the students' minds. For example, the following shapes might be shown:



The students would be led to see that shape A is not round because it is not closed; that shape B is not convex; and that shape C is not differentiable.

Obviously, the two kinds of roundness - everyday and mathematical - are very different in the mind of the student. Most students would never be able to offer a suitable definition for what they easily recognize as roundness in everyday life; and they would probably never guess that the mathematical definition as stated above applied to the word 'round'. They would not hesitate to state that some objects are 'sort of round' or that one object is 'more round' than another. Such language is not acceptable in mathematics.

Furthermore, the mathematical method of learning the concept does not adhere to the theory of prototype concept discussed above. If a young child held the prototypical view of roundness as perfectly circular, that child might apply the word 'round' to a square, since a square resembles a circle, for instance, in its closedness, or in its regularity. Even if the teacher pointed out why squares are not round, and even if the child may come to realize that the word 'round' is not to be used in connection with the object, still the similarity has been recorded in the child's

mind, and the square remains attached peripherally to the 'round' concept. A similarity, once seen, cannot be unseen.

Finally, mathematical language is very precise about assigning words to special cases of concepts. For example, a circle is a special case of an ellipse. But if you show a circle to a child and ask whether it is an ellipse, the child is likely to say, "No, it is not an ellipse; it is a circle." This is probably because he has learned the concept of ellipse in the usual way, that is, he has seen many shapes which have been called ellipses and has assigned the linguistic tag to shapes which resemble those he has seen (perhaps as a prototype concept). But in his experience, he has never heard a circle called an ellipse. If language is seen as a communicative tool, then the best way to describe a circle is to use the tag 'circle' and not 'ellipse', since the word 'ellipse' probably communicates a non-circular shape.

The point of drawing all these distinctions between mathematical concepts and 'everyday' concepts is not only to demonstrate that what goes on in the mathematics classroom can be very different from children's out-of-class experience. It is also intended to illustrate how mathematics can seem artificial to children, since the concepts they learn in school are different *in kind* from those they learn outside school. Small wonder they find mathematics uninteresting if they cannot see mathematical concepts as tools for coping with reality.

Concepts must arise from a need

One of the skills of mathematical thinking is to take an everyday concept and introspectively to analyze one's own thoughts to arrive at a more or less mathematical definition. The objective is to be able to apply that finely honed definition productively to new situations. For example, we might take the everyday concept of 'even number' which children can be expected to know, and through analysis eventually arrive at a definition such as "a number of the form $2n$, where n is an integer". We could then apply this definition productively in answering such questions as, "Is 0 an even number?"

That is basically what mathematics is about: abstracting from real life situations, deducing properties from the abstractions, and applying those properties to other real life situations. That is the argument given to students for why they are studying, for example, imaginary numbers. The system of

complex numbers arises from a need, namely the failure of the real number system to solve certain equations, and through the deduction of certain of their properties, complex numbers can be applied to many situations where both the equation and its solution contain only purely real numbers. The general solution to the differential equation

$$y'' + y = 0$$

can be found by using complex number methods, as

$$y = Ce^{ix} + De^{-ix}$$

even though the final solution can be written in real numbers as

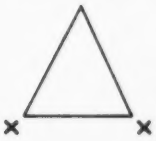
$$y = A \cos x + B \sin x$$

The solution has dipped into the realm of imaginary numbers, which are useful in arriving at a real solution.

If the message that concepts, including mathematical ones, arise from a need were made clear to students, they might find mathematics much more interesting and relevant. As it is, much of our mathematics teaching omits both the head and the tail of this process. At the head, students are not shown *why* a certain classification has been invented. They are simply presented with an already-formed definition and asked to analyze whether certain objects or sets of objects are members or subsets of the defined class. At the tail, students are not shown the applications for which the definition was constructed, and thereby do not tie mathematics to reality.

This deficiency in mathematics teaching shares some features with language teaching. If we adopt the view of language as a tool for communicating (either with one's self or with others), we see that vocabulary study and sentence parsing neglect the goal of making one's language more communicative. If students felt a need for a concept in order to communicate some idea, they might see the usefulness of inventing words for concepts. If students could first be asked to describe, for instance, a figure to classmates who could not see the figure but who were asked to draw a reproduction of the figure from the description, both describers and drawers might find certain terms useful. An example is cited by Wight (1983) of a student who is asked to describe the following figure to a classmate:

Actual figure



Attempted drawing

When the classmate continually drew the figure incorrectly, the describer could only tell him that his *x*'s were not down far enough. After several lowerings of the *x*'s, the drawer finally placed them at the vertices of the triangle. But if the describer had known the word 'vertex', the exercise would have been simpler. Both students would appreciate the usefulness of the concept of vertex. And the describer would have gained communicative competence in the sense that he could henceforth use language more efficiently to describe a given situation.

One major difficulty with the above discussion is the assumption that an everyday concept can indeed be analyzed and broken into certain components which are common to all members of the class under discussion. For example, it was possible to isolate those properties of roundness which are common to all round objects and only to those objects. The objective is to get to the essentials of the classification. In practice, though, real life concepts may not have a core of essentials. The notion of a dog is clear to most people, but if we try to isolate the essential attributes of dogs, such as four-leggedness or barking, it will always be possible to produce creatures which are clearly recognizable as dogs but which lack some essential attribute. Three-legged dogs can be found, as well as dogs which do not bark.

Monothetic and polythetic concepts

Concepts which have a central core common to all elements of the class are often called *monothetic* concepts, as opposed to *polythetic* concepts, which are based on various resemblances. In a polythetic concept, element A may resemble element B in one way, element B may resemble element C in another way, while elements A and C may not resemble each other at all, and yet all three elements belong to the same class.

Everyday concepts are very often of this polythetic nature, while mathematical concepts are not. A 'social' language will

possess words with multiple meanings which at one time may have been related, but which have lost their link. The slang expression "he's a real square" may have originally had a connection with the geometrical shape, but it would be difficult to reconstruct the origin of the expression. Thus, 'square' is a polythetic concept.

It is virtually impossible to define mathematically a polythetic concept. This fact presents a real dilemma for mathematics, for the process of analyzing a real-world concept and creating a mathematical definition for it, as discussed above, becomes impossible in most cases. For the student, words for mathematical concepts are therefore quite different from words for everyday concepts.

The result of this line of thought is that students, upon hearing a word like 'smooth' in a mathematics class, must decide whether it is a mathematical word or an everyday word, because the way that word is treated depends on that choice. If the word is taken in its mathematical sense, then the mind must process it in a mathematical mode by appealing to the precise definition. In this way, we have returned to the distinction raised in Chapter 1 between the mathematical 'register' and everyday register. Children must figure out which register is being used, as in the example of Chapter 1, where a child was confused between the mathematical term 'diagonal' as joining two vertices of a polygon, and the everyday 'diagonal' as meaning 'slanted, not vertical or horizontal'.

The mathematics teacher can provide help in overcoming this confusion by guiding the student to see that the mathematical concept is not the same as the everyday word, but that there is some connection which may serve as a mnemonic device. Thus, 'round' can be rememberd as having something to do with everyday roundness, but in the mathematical sense, it must be treated differently by the brain.

Physicists have attempted to solve this problem by assigning common names to concepts quite arbitrarily. In the study of fundamental particles, a difference in behaviour may be noted between two particles previously thought to be identical. Regardless of the nature of the behaviour, the particles are assigned some attribute, such as 'colour', or even 'charm'. The label in no way suggests that the particles are in any way charming; it simply groups them in a category because they behave alike in some way.

Nowhere are the problems associated with polythetic concepts illustrated more clearly than in the distinction between

perceptional concepts and functional concepts, that is, between classes of objects which resemble each other, and those which perform the same function. For example, 'quadrilateral' is a perceptional concept because of the way quadrilaterals look, but 'translation' is more of a functional concept because of what translations do to figures. But consider the word 'star'. The six-pointed 'star' of David is called a star because its shape somehow resembles that of a twinkling star in the sky. But a movie 'star' earns the same name because he/she somehow shines. Now in reality the star of David and a movie star have nothing directly in common except their label, and yet indirectly they are related via the stars in the sky, one related to how stars look, the other related to what stars do. 'Star' is therefore a polythetic concept which would be difficult for mathematicians to define.

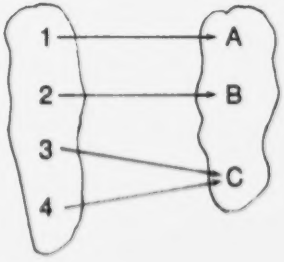
If one were to ask teenagers what Elvis Presley has in common with a five-pointed object, they may see the connection and answer "They are both stars". In the sense that they have something in common, both Elvis and the five-pointed object are part of the same concept, and that concept is monothetic only in the sense that all of its members share the same word.

The concept of 'function' learned by most students is mathematically equivalent to the concept of 'mapping'. And yet there is a tremendous confusion in students' minds (as well as teachers' minds) between the two concepts. Why, indeed, should there be two words for the same concept? The obvious reason is that 'function', defined as a set of ordered pairs no two of which have the same first element, is a perceptional one, since it is easy to visualize an ordered pair such as (5,7). Mapping, in the sense of drawing an arrow from each element of the domain to one in the range, is basically a functional concept, since it represents an action performed on an object.

Function

{ (1,A), (2,B), (3,C), (4,C) }

Mapping



If teachers can let students realize that the same word can refer to both a functional concept and a perceptual one which are in fact identical, much of the confusion between function and mapping might be eliminated.

Actually, the ability to view an object as belonging to many different classification schemes is closely related to lateral thinking, and is an important skill in mathematics education. Being able to place an item in several category schemes simultaneously appears to be a developmental skill, according to Piaget and Inhelder (1956). They and others have noted that younger children focus on only one class aspect at a time. It is only after they have attained the formal operational stage that they may consider objects or classes of objects from several points of view.

Teachers can aid children in developing these sorts of skills. Sinclair (1983) describes a useful game which children enjoy playing. The children are shown the partially filled table as pictured below, and they are asked to provide possible words which might fill in the remaining boxes.

seagull		lighthouse
swan	barge	
		cathedral
	tractor	

For example, the children may notice that the first line, containing 'seagull' and 'lighthouse', concerns the sea, while the second column seems to contain modes of transportation. They may therefore decide to put 'ship' in the row one, column two position. Sinclair goes on to say that if concepts are to be part of language as a communicative tool, it is important that the task should be done by groups of students, so that full discussion can occur. In this way, the students will have to verbalize their classification

schemes, and their concept formation will be done in the context of communicating ideas with their classmates.

Levels of abstraction

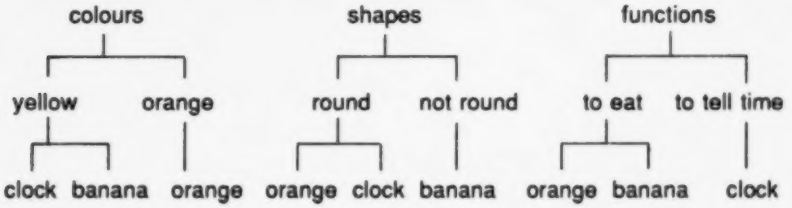
The examples of concepts dealt with so far in this chapter have been relatively simple ones, such as colours, shapes, and sizes. Mathematics deals with more complex and abstract concepts. What does it mean to say that one concept is more abstract than another? For the purposes of this discussion, let us focus on the hierarchical nature of concepts, that is, the notion of classes of classes. For example, children must learn the concepts of 'red' and 'blue' before grouping those abstractions into the concept of 'colour'. In this way colour is higher in the abstraction hierarchy because the objects classified as colours are themselves already concepts.

Similarly, shape and size are higher order concepts, since various shapes such as circles and rectangles are already abstractions. We may even go to a higher level of the hierarchy by grouping colour, shape, and size into the category of 'physical attributes'. In mathematics, hierarchies can become quite complex, piling up many levels of abstraction, and culminating in such notions as category theory, which groups together already highly abstract concepts such as groups, vector spaces, or topological spaces, into the higher order abstraction of 'categories'.

Jerome Bruner (1973) argues that language is especially important in labelling these higher order concepts, since it facilitates the transfer from one category into another. When a child sorts attribute blocks and states, "I did it by colour", he/she is demonstrating an awareness of other criteria for selection, for example by shape or by size. Use of the term 'colour' is a manifestation of a sophistication in thinking which is not apparent in a statement such as "I put the red ones together". It is the role of the teacher to foster the use of higher order conceptual words, according to Bruner, in order to develop the ability to transfer from one classification scheme to another.

Bruner and his associates performed several experiments in which they asked students to classify objects and to explain their choice of classification. This was carried out with Wolof-speaking students in Senegal, as well as with French children.

Bruner describes one possible representation of the hierarchical structure of his experiments' concepts as follows:



Bruner's results confirmed that the use of higher order terms correlates highly with the tendency to classify according to different attributes:

> Our results do indeed indicate that there is a significant association between superordinate words 'color' and 'shape' and the number of different types of attributes used for grouping. And this relationship holds when all other factors such as knowledge of French and school grade are held constant. Thus, if a Wolof child utilizes a superordinate word, his chances of grouping by a variety of attributes are twice as great as those of a child who utilizes no superordinate vocabulary. (p. 43)

This experiment was carried out with Wolof children in Wolof, with Wolof children in their language of instruction (French), and with French children in French. The relationship between superordinate order words and the variey of classification was strongest when the subjects used their first language, especially the French group. Bruner speculated that the Wolof students operating in French may use the superordinate language in a 'forced' way without truly having mastered its use. He concludes that "access to the pure conceptual hierarchy is indicated by the use of abstract terms *only if* the linguistic terms have been thoroughly mastered in all their linguistic implications."

The one flaw in Bruner's logic is that he has somehow assumed a causal relationship between higher order vocabulary and the ease of transfer between classification systems. The evidence he produces is merely correlational, as is evidence presented by other researchers. The relationship could be other

than causal, for example, if children who were adept at using various classification schemes naturally used a vocabulary which described their thoughts.

We have seen earlier that it is difficult to state whether language precedes thought or vice versa. Likewise, it is difficult to state whether higher order vocabulary precedes hierarchical thought or vice versa. It can only be said that the experimental evidence supports Bruner's claim that there is some kind of relationship, but that the nature of that relationship is likely to be quite complex. Practically speaking, however, we have seen that language, while perhaps neither always a direct cause nor a direct effect of conceptual thought, can be a useful aid in mastering concepts. Bruner is therefore probably correct in surmising that teachers may be able to use higher order abstract terms in order to promote the mastery of hierarchical concepts.

But in light of our earlier discussion, making concepts fulfill a need is perhaps the key to both hierarchical thought and to higher order vocabulary. It makes little difference whether the thought precedes the vocabulary or vice versa; both must be prepared by advance organizers which make the student see a need for hierarchical thought as well as for higher order vocabulary in order to attain a computational or a communicative objective. Once that need has been created, the child will naturally pick up the new term or concept as the tool he has been looking for.

Syntax

This discussion has dealt so far only with the vocabulary aspect of language. Is there any relationship between concepts and the grammatical or syntactical aspect of language? Again, it is Bruner who provides experimental evidence to the affirmative. In his studies with the Wolof in Senegal (Bruner *et al.*, 1966), children's descriptions of their classification strategies were recorded, and the results compared with the tendency to classify according to multiple schemes. Some of the children were unable to verbalize their strategies at all and simply pointed at certain objects. Others were at least able to come up with an isolated word or two, such as 'red'. Still others were able to formulate their response in a complete sentence, such as "They are all red". The subjects in this last group were far more likely (four times as likely) than the label-only group to use multiple classification schemes. The conclusion was that the formation of complete

sentences is related to the ability to form hierarchies of concepts. The use of labels alone was helpful, as the children who used isolated words were superior to those who did not, but the use of grammatical sentences provided an even stronger aid to conceptualization.

But here again, Bruner seems to be hinting that the relationship is a causal one. He wants the reader to believe that by teaching grammar, it is possible to heighten children's skill at conceptualization. Perhaps the reverse is true: that if children learn to conceptualize better, they will tend to verbalize in complete sentences.

The use of complete sentences is often connected with the ability to write. It is therefore natural to ask whether written language is a major factor in conceptualization. Experimentation in this area was done by Vygotsky (1961), who claimed:

> Our investigation has shown that the development of writing does not repeat the developmental history of speaking. Written speech is a separate linguistic function, differing from oral speech in both structure and mode of functioning. Even its minimal development requires a high level of abstraction. (p. 98)

Putting thought into writing is a distancing action, in which the word disengages itself from the mental concept. In fact, Vygotsky draws the analogy that writing is to oral speech as algebra is to arithmetic, since in writing, "words are replaced by images of words" in the same way that numbers are replaced by algebraic symbols for numbers.

It is this same dissociation of thought from word that makes the study of grammar important to cognitive development. It is not the conjugation of verbs or the parsing of sentences in isolation which is important for education; it is rather the training of the mind seemingly to step outside itself to analyze its own language. This analysis applies equally to the structure of one's own language as well as to its function. This skill is one of the principal skills of mathematicians.

The difficulty for teachers as well as for researchers is that even if it were well documented that training in writing or in grammar has definite effects on psychological functions, it is not true that the effects are proportional. One cannot expect that every x hours of training in one area will lead to an increment of y

in cognitive skills. Progress is uneven. Vygotsky noted that this same unevenness applies to mathematics learning:

> The different steps in learning arithmetic may be of unequal value for mental development. It often happens that three or four steps in instruction add little to the child's understanding of arithmetic, and then, with the fifth step, something clicks. The turning points at which a general principle becomes clear to the child cannot be set in advance by the curriculum. The child is not taught the decimal system as such; he is taught to write figures, to add and to multiply, to solve problems, and out of all this some general concept of the decimal system eventually emerges. (pp. 101-102)

Vygotsky's arguments fly in the face of programmed learning and behavioural objectives, which promote the view that children can pick up concepts one at a time. He would certainly have argued that behavioural objectives trivialize the curriculum into such a low order of conceptualization that meaningful higher order concepts are neglected. It is tempting to use programmed textual material because the students' progress seems to be apparent at every step, as they learn the small bits of material one by one. But perhaps we are fooling ourselves if we concentrate only on quantitatively measurable progress at the expense of the global cognitive development of the child. It might be better to aim at instilling higher order processes even if we cannot be sure if and when we have succeeded.

Summary

There are different forms of concepts, and they can be learned and formed in different ways. There is no ideal method for teaching all types of concepts. Teachers must be aware of the complexity of the issue and avoid the dogmatism of claiming that all concepts should be taught to all students in the same way.

Teachers should also be aware that real life concepts and mathematical concepts may be very different, and that students may learn them in different ways. When using a word, care must be taken that students understand in just which 'register' (mathematical or otherwise) the term is being used.

If Bruner's theory of cognitive development is correct, teachers should also encourage the use of higher order abstract terms in children's vocabulary, as well as the use of complete grammatical sentences.

Finally, students need to place mathematics in a context: they need to see where concepts come from and where they are leading. They must experience the need to use a concept in their own language.

We have seen that although there is no precise one-to-one correspondence between language and concepts, they are very closely related in very complex ways. Mathematics teachers cannot claim that they are merely teaching mathematical concepts and not language. Teachers of mathematical concepts are very much language teachers.

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CHAPTER 4

LOGICAL CONNECTIVES

The use of logical connectives such as 'and', 'or', 'if....then' is an exercise in the formation and manipulation of concepts. For if A and B are concepts in the sense of sets of objects, then their union and their intersection are also concepts. The negation of a concept is the complement of the set of objects in that set, so that 'not A' is a concept in its own right. Much of what was said in Chapter 3 can be applied to the use of logical combinations of concepts.

Negation

In mathematical reasoning, the complement of a set is defined by the set itself, so that by defining the members of set A, we automatically define the members of ~A. Linguistically, most languages express negation quite simply; for example, the English language merely adds the word 'not'. In principle, therefore, it ought to be easy to reason with negatives. There is substantial evidence, however, that such is not the case.

One of the early and well-known discussions of negation and other logical connectives was undertaken by Miller (1969), who pointed out:

> In effect, negative sentences are psychologically more complicated, because they require all the cognitive processes normally involved in understanding an affirmative sentence, plus some further process of denial or negation.

In practice, people seem to prefer expressing the positive, even if it is more complicated than the negative expression. For example, a study by Wason (1959) asked subjects to respond to the truth value of statements such as "Four is an even number",

"Nine is not an odd number", etc. He measured the time necessary for the subjects to respond, along with their success rates, and found that negative sentences took considerably longer to process, and resulted in significantly more errors than did positive sentences. Responses to true statements were not significantly different from responses to false statements in either time or correctness. In a similar experiment, subjects were asked to fill in a blank in a sentence such as

_____ is not an even number.

The existence of the word 'not' produced significantly slower responses together with a higher rate of errors.

Many experiments similar to the one just described have yielded the same type of results, all indicating that the processing of negatives in a sentence is a hurdle for logical reasoning. The experiments also indicate that it is not just the addition of an extra word which renders the task more difficult; it is the negative *meaning* which is the major hurdle.

Teachers and writers of textbooks and other materials should therefore take note of the number of negatives they are using. They should realize that a negatively formed sentence may be more difficult for students to handle than a positively formed one. This statement applies doubly in the case of double negatives.

Conjunction and disjunction

A conjunctive concept represents the intersection of two concepts in the set-theoretical sense, while a disjunctive concept (in mathematics) is the union of two concepts. For example, if a child has learned the concepts 'round' and 'red' then the conjunction of the two concepts is the set of all objects which are both round and red; the disjunctive concept is the set of all objects which are either round or red (or both).

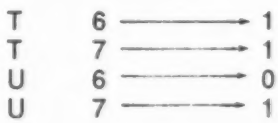
What is the role of language in the formation of such concepts? Do children naturally verbalize the phrase 'round and red'? One test for measuring concept formation is the 'four-card problem'. In this test, subjects are shown two pairs of coloured cards and are told that only one pair is correct according to some rule that the experimenter is thinking of. (This may be a conjunctive rule, such as 'the pair with both a red card and a green card' or a disjunctive one, such as 'the pair with either a red card

or a green one', but the subject is told none of this. The subject is asked to guess which pair is the correct one, and is told whether the guess is right or wrong. Another two pairs are shown, another guess made, followed by reinforcement. After several attempts, subjects should be able to guess repeatedly the correct pairs as evidence that they have mastered the concept. After mastery (defined as ten correct hits in a row), subjects are asked to describe the rule.

The experimental evidence (Gay, 1967; Zepp, 1983b) shows clearly that the vast majority of subjects are unable to describe their rules. Gay reports such answers as "God told me" or "I just did it", but almost never an analytical response such as "It's the red one and the green one." Thus, it appears that conjunctive and disjunctive concepts are usually formed before they are verbalized.

The odd fact about the four-card problem is that greatly differing results are obtained from speakers of different language groups. Some groups find the conjunctive concepts easier than the disjunctive ones, while the reverse is true with other language groups. It is still not clear what features of the various language groups affect performance on the test. These curious results would not be expected if the concepts are formed without the use of language. It leads one to hypothesize that thought is conditioned by the particular language one speaks. This hypothesis, known as the Whorf hypothesis, is discussed in detail in Chapter 7.

As we have seen, mathematics learning entails more than the mere acquisition of concepts; the conscious verbalization of those concepts is in itself an important objective in mathematics education. The question arises, which kinds of logical concepts are easier to verbalize, and why? One interesting experiment on this topic was done by Bruner (1966). The problem was disguised as the interpretation of results of a chemistry experiment, in which subjects were asked to describe which combinations of chemicals gave the desired result. They were told that either chemical T or chemical U could be mixed with either chemical 6 or chemical 7, and the results were shown in one instance as follows, a 1 representing the successful production of the desired new chemical, a 0 denoting failure:



Thus, mixing chemicals U and 6 failed to produce the desired result, while all other combinations were successful. It can be seen that the presence of either chemical T or chemical 7 (or both) produced the desired result. Subjects were marked according to both the accuracy and the conciseness of their responses. They repeated the experiment for each of 6 different tables, which represented the concepts T or 6, not-T and not-6, not-T or 6, T and not-6, T or 6 but not both, and T and 6 or neither. Figure 4.1 shows the mean scores for the six concepts.

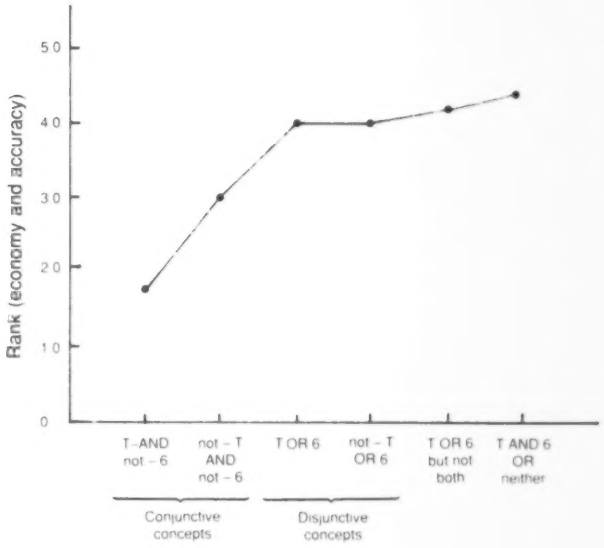


Figure 4.1. Mean scores for six concepts. (Bruner, 1965)

Thus, the conjunctive concepts were easier to master than the disjunctive ones, even when the added difficulty of negation was present. As we might predict, the subjects had a great deal of difficulty with the more complicated tasks, and tended to give descriptions which focused on positives rather than negatives. Subjects tended to overlook simple negative descriptions such as "the absence of chemical T gives the desired result".

Mathematics teachers will immediately have noticed that the tables shown in the experiment resemble truth tables. There may be some feeling that truth tables are a way of making logic easier for students. Truth tables are included in courses in logic in the hope that they will produce more logical thinking. The poor performance of subjects on Bruner's experiment indicates the contrary. Bruner concluded:

> Perhaps the logically naive individual does not think in a logically pure and truth-functional way.

The final question in this vein is how easily conjunctive and disjunctive concepts, once acquired and verbalized, can be applied to deductive situations. Zepp (1982, 1987), showed subjects cards with colours on both sides,and asked them to make possible deductions about the hidden side of the card based on the shown side and a statement. For example, if a red side A is shown and the statement, "Side A is green or side B is black" is read, then the correct deduction is that the hidden side B must be black.

Secondary school subjects, as well as university students, found the test difficult, particularly for disjunctive concepts. As expected, the addition of a negative (e.g., "Side A is green or side B is not black") rendered the task more difficult, and two negatives ("Side A is not green or side B is not black") made it still more difficult.

An interesting incorrect response to disjunctive items was frequently made if the card shown seemed to contradict the truth value of one of the propositions of the statement. For example, if the statement is "A is green or B is black", and a red A is shown, subjects may think an error has occurred because the red A seems to contradict the "A is green" part of the statement. The correct response would be that if at least one of the propositions connected by 'or' is true, then when the first is shown to false, the second must be true, and so in this case B must be black. The fact that so many students encounter this difficulty may be one of the difficulties in indirect reasoning or indirect proofs.

Exclusive disjunction

In everyday language, the English word 'or' has two meanings, an inclusive one and an exclusive one. To the question, "Would you like some tea or some biscuits?", it is natural to

respond, "I would like both," while to the question, "Would you like coffee or tea?", a response of "both" is certainly not expected. In most instances, the English word 'or' is used in the latter sense, that is, in the exclusive sense, meaning "one or the other but not both." In mathematics, on the other hand, the word is given its inclusive sense; when we describe the union of two sets A and B as the set of elements in either A or B, we need not add "or both". The curious fact is that the concept of union, one of the principal foundations of all mathematics, does not have any unambiguous term in the English language. I have examined many other languages around the world (e.g., Zepp, 1983a), and have yet to find a language which has a clear word for the inclusive 'or' as distinct from the exclusive one. This seems to indicate a real difference between the foundations of mathematics and the foundations of language development. This point is taken up again in Chapter 12.

This dichotomy can be easily demonstrated experimentally. The acquisition of exclusive disjunctive concepts as measured by the four-card problem proves a virtually impossible task (Zepp, 1983b), while the use of the exclusive 'or' seems quite natural and frequent in most languages. In any event, mathematics teachers should be well aware of this difference, and should point out to their students the ambiguity of the word 'or' in everyday speech as opposed to its precise inclusive meaning in mathematics.

Implication

The implicational statement "If A, then B" is a compound concept just as conjunction and disjunction are. It has the same truth table as (and is therefore logically equivalent to) the statement "B or not-A". Students in introductory logic classes usually have a great deal of difficulty understanding this equivalence. The two-sided card experiment described above shows that responses to the same card are radically different depending on the wording of the statement. For example, a statement like "A is not red or B is green" is logically equivalent to "If A is red, then B is green", but the two statements appear to be processed psychologically in very different ways, as evidenced by the very different response patterns. If the wordings are logically equivalent, and easily transformable from one form to the other *a la* Chomsky, why should the responses be so different? It

certainly appears that logical equivalence is not at all the same thing as psycholinguistic equivalence.

The answer may lie in the discussion of set-theoretical concepts *vis-a-vis* functional concepts (Chapter 3). The wording "if A then B" may be treated functionally as meaning that A causes B, or perhaps precedes B, while the alternate wording "B or not-A" suggests a set-theoretical treatment. The two types of concepts may be mentally processed so differently that students may not even be able to appreciate that the two are logically equivalent.

Think of the ways in which 'if-then' phrases are used in children's lives. They invariably are associated with either causal situations (A causes B) or temporal ones (A occurs before B). This is not the case in mathematics; for instance, the sentence, "If two lines are parallel, then the alternate interior angles are equal," is neither causal nor temporal, and might be termed 'pure' logic. Thus, the 'if-then' statements of mathematics classes may be very foreign to children's actual experience.

Research has supported the view that logic in cases where the propositions can be related either causally or temporally is easier for children than in cases of pure mathematical logic. For example, Johnson-Laird (unpublished, reported to Bruner, 1965) questioned subjects on tables similar to the chemical experiment tables reported above. In some cases, the propositions contained related material, such as "the engine requires oil" and "the oil pump is faulty"; in other cases, worded almost exactly the same, the propositions were not related, e.g "the engine requires oil" and "the telephone is faulty." The subjects' responses were categorized as to whether they were logically pure, or whether they contained causal or temporal elements. The results did not provide conclusive evidence, as no significant differences in accuracy or conciseness were observed between the two types of statements (related versus unrelated). However, it generally took considerably longer for subjects to respond to unrelated items, and the types of explanations were different. As the number of positive instances increased, the subjects used fewer purely logical explanations in favour of more causal explanations for the related items, and in favour of temporal explanations for the unrelated items. Bruner's assessment is as follows:

> Individuals do not naturally engage in truth-functional thought. They are always ready to leave the logical requirements of the task behind and to try to establish some meaningful connection between events. If the

> events cannot be directly related, the best that can be done is to establish some temporal connection between them. But where the events do seem to be related to one another, the subjects will spend a considerable amount of time attempting to derive a satisfactory causal model of them. (pp. 81-82)

The temporal view of inference causes difficulty when reasoning from the contrapositive (the contrapositive of "if A then B" is "if not-B then not-A" and is logically equivalent to the original statement). Consider a causal/temporal statement such as "If it rains today, I"ll stay home." The correct contrapositive statement is taken after after the fact: "If I went out, it must not have rained," or else, "If I go out, it will be because it has not rained." It would *not* be logically equivalent to state, "If I go out, it will not rain." Thus, reasoning from the contrapositive usually proves very difficult for students, not only because it involves a double negation, but also because it requires a reorientation in time.

Research shows that students reason much more readily from the converse than from the contrapositive (e.g. Zepp, *et al.*, 1987). The converse of "If A then B" is "If B then A" and is not logically equivalent to the original statement. Thus, reasoning from the converse is often incorrect reasoning. For example, in the hidden card experiment described above, if students are shown a green side B and the statement is "If A is red then B is green", they all too readily jump to the conclusion that side A must be red. Although this response in itself is no proof that converse reasoning must have been used, we can speculate that the subjects are assuming that the original statement is equivalent to "If B is green, then A is red", i.e., the converse of the original statement.

The same difficulty with time inversion occurs with stating the converse of a statement. The converse of "if it rains, I"ll stay home" is "If I stay home, it will be because it has rained", in which case the speaker must orient himself to a hypothetical situation in the future and conclude something about the past. We might expect, therefore, that students would be reluctant to reason conversely, just as they do not readily use the contrapositive. Why is it, then, that students are willing to reason conversely (and incorrectly), but not contrapositively (and correctly). No doubt the double negation is a great obstacle. But is it the only obstacle?

We might investigate whether students readily reason from the inverse statement (the inverse of "if A then B" is "if not-A

then not-B", which requires the negation of both propositions, but no inversion of hypothesis and conclusion.) The hidden card experiment shows that if the statement is "If A is red, then B is green" and if a black (i.e., non-red) side A is shown, many subjects will conclude that side B is not green. We may again speculate that for the most part they are assuming (incorrectly) the equivalence of the statement with its inverse ("If A is not red then B is not green"). The frequency of responses of this nature is slightly less than that of converse reasoning, but far higher than that of contrapositive reasoning. Thus, students are in fact willing to make the double negation of the inverse. It is only the two-step process of double negation together with inversion which they are reluctant to consider.

The hidden card test items give no indication of cause - side A does not 'cause' side B - and very little indication of time - A does not necessarily come before B chronologically, although it does alphabetically. The logic required tends to be of the 'pure' variety, which, as we have seen, can be quite foreign to students. It may be that because the experiment was done in their mathematics classroom, they were looking for some rule or formula, and latched onto the converse and inverse transformations, applying them blindly without actually connecting them to reality.

A second possibility is that in real life, children grow up mostly hearing 'if-then' statements with biconditional ('if and only if') meanings. For example, if the mother says, "If you are a bad boy, you will get a spanking," the child takes the sentence to mean that if he is a good boy, he will not get a spanking. In the strict mathematical sense, however, he could be a good boy and still get a spanking. Thus his real life experience would lead him to believe that converses and inverses are equivalent to the original 'if-then' statements.

To investigate the effects of mathematical language on the reasoning process, the hidden card experiment was repeated (Zepp, 1987) with similar subjects in a similar setting, but this time the cards had mathematical graphs of functions instead of coloured squares, and the statements were of the form, "If function *f* is differentiable, then function *g* is continuous." As might be expected, the addition of mathematical terms and graphs made the task more difficult for many subjects. A few displayed a certain 'mathematics phobia' and missed even the easy items. What is more interesting is the fact that many students who correctly answered easier items missed the harder items in different ways

from those students who had taken the test with coloured cards. Instead of using a consistent (but perhaps incorrect) strategy of converse reasoning or inverse reasoning, the mistakes appeared to reflect more random guessing. Thus, students who might have pursued a consistent pattern of logical thought in the case of coloured cards, reverted to confused guessing or some other inappropriate strategy when mathematical content was introduced.

The results indicate a certain cognitive overload in some cases. A student can cope adequately with a certain number of obstacles, but once a certain threshold has been reached, consistent strategies are abandoned in favour of confused guessing. In this manner, a seemingly small obstacle (such as a mathematical term), which students can easily cope with in isolation, becomes the 'straw which breaks the camel's back' when coupled with other obstacles, resulting in mental overload and confusion. This is why mathematics teachers must choose their language carefully: it may appear that a student understands a given word in isolation, but if he is at all unsure of its meaning in an applied context, that lack of confidence may be the single element which pushes him over the threshold from understanding to total confusion. When advancing students from one level of understanding to another, teachers must gauge the size of their steps so that students feel comfortable and confident in reaching for the next step.

Comparisons

Expressions like 'more than', 'at least', and 'not less than' can confuse students. An often-used test for investigating how students deal logically with comparisons is the 'three-term series problem', in which subjects are given two statements such as

John is shorter than Fred
John is taller than Charles

and are asked what conclusions, if any, can be drawn. In the view of some theorists, students solve all such problems using some form of mental imagery. For example, they may picture John standing next to the taller Fred on one side, with a shorter Charles on the other side. In such representations it is easy to see that Charles is the shortest, with John in the middle, and Fred the tallest.

Huttenlocher (1968) has elaborated this theory of spatial images by stating several principles which lay guidelines for the way in which images are formed. For example, she claims that the image 'end-anchors' on the subject of the first statement. For example, if the premises are

A is taller than B
C is shorter than B

subjects will quickly anchor on A above B, then will place C below B, and will readily conclude that A is taller than C.

The spatial image theory should be relatively free of linguistic constraints, for the images are not language-oriented. However, it is easy to show that the choice of wording has a large effect on the results of the three-term series task. Clark (1969) argues that difficulties in solving such problems are to a large extent linguistic in nature. He sorts these difficulties into those due to various linguistic principles, e.g.:

1. The principle of lexical marking

This principle states that supposedly opposite pairs such as tall-short are assymmetrical: people tend to focus on one pole more than the other. For example, we may ask, "How tall is John?", even if John is short; but we would not ask, "How short is John?" if John were tall. According to this principle, the more positive pole (in this case, tall) is processed and stored by the mind in a simpler, more accessible way than is its opposite. Thus, test items using the word 'shorter' should be more difficult than those using the word 'taller'.

2. The principle of congruence

This principle states that once the brain enter one train of thought in the direction of one of the poles, it is hard to reverse direction and consider the opposite. Once the direction, 'tall' is embarked upon, an additional linguistic step is necessary to focus on shortness. Thus, the test item stated above is rendered more difficult because the use of the assymmetric pair taller-shorter adds an extra step to the linguistic processing. If the word 'taller' had been used in both statements, the item would have been much easier.

Clark tested the hypothesis that these two principles can contribute to difficulty in solving the three-term series problem, by measuring the time necessary for subjects to solve such exercises. The series used variations of the words 'good' and 'bad', such as

John is better than Fred
John is worse than Charles
Who is best?

The principle of lexical marking was supported by the fact that when the question was "Who is worst?", subjects took considerably longer to answer it than when it was worded "Who is best?" The principle of congruence was also supported in some instances, but not all. In fact, the item which required the shortest mean time to solve (4.84 seconds) took the form, "C worse than B; A better than B; Who is best?". We would have expected the use of both poles of the better-worse pair to have been an additional obstacle to subjects, but apparently it was not in this case. The principle of congruence showed up most clearly when both of the major premises were congruent but when the question varied. For example, the wording "B worse than A; C worse than B" had a short mean solution time of 5.04 seconds when the question was congruent with the premises, "Who is worst?", but took a much longer 5.93 seconds when the question was not congruent: "Who is best?".

Finally, mean times soared into the 6 and 7 second range when the questions used the phrases "not as good as" and "not as bad as". The item "B not as bad as C; B not as good as A; who is worst?" took 7.12 seconds to solve, on average. This result is evidence that the linguistic processing of such a phrase may be a serious obstacle to understanding the logic of the situation.

Other linguistic variables

Studies have been carried out to determine the effect of various linguistic variables on logical inference. One such study was that of Jurdak (1974). Twelve separate linguistic variables were considered, such as the number of noun phrases or verb phrases in the item, existence of the word 'all', the number or position of negations, or the total number of words. The test was given to Arabic/English bilingual students in grades 6 to 10, and

the results were subjected to the statistical procedure of stepwise multiple regression at each grade level and for each language. This techinique is used to determine which variables (regressors) most strongly affect logical performance. Results were not consistent from year to year, but in the overall results, the existence of the Arabic equivalent for the word 'all', the number of transitive verbs, and the position of the negation were all significant regressors for the Arabic group. For the English group, significant regressors included the position of negation, the number of transitive verbs, the total number of words, the number of articles (a, the, etc.), and the number of verb phrases in the item. Variables which were not significant regressors included the numbers of nouns, noun phrases, possessives, adjectives, conjunctions, and inversions in the items. It appeared that the linguistic variables had a stronger effect on performance for the English group than for the Arabic group, a result which is not surprising since English was the second language of the Lebanese students. Perhaps surprisingly, the linguistic factors appeared to have a stronger effect for the older children than for the younger ones in the English group, but not for the Arabic group. Jurdak offered the explanation that "perhaps syntactic structures reach stability in a foreign language later than those of a native tongue".

Another interesting result of the Jurdak study was that the position of negation appeared to have a stronger effect than the actual number of negatives in the sentence. This supports findings of Roberge (1969) that negation in the major premise of a logical syllogism is an important factor in predicting difficulty in forming conclusions.

Even though the above results were not consistent from grade to grade or from language to language, the fact remains that at all grade levels, linguistic variables, both in Arabic and English, did contribute to the difficulty in assessing logical propositions. Jurdak concluded:

> Decisions made by children concerning the validity of inferences do not seem to be made solely on a formal basis but also on the kind of linguistic structures of the natural language used. (p. 237)

An earlier study by Jansson (1974) produced similar results.

In both the Jurdak and Jansson studies, quantifiers such as 'all', 'not every', or 'none' led to difficulties in the subject's logic. It is not immediately apparent why such difficulties should arise, for the language seems simple enough. One early explanation used the concept of 'atmosphere effect'.

Woodworth and Sells (1935) argued that the positive or negative wording of a syllogism creates an 'atmosphere' which affects the subject's response in directions consistent with that atmosphere. In practice, this means that if one of the premises stated is negative, subjects will tend to give a negative conclusion. Likewise, if the word 'some' occurs in one or both of the premises, subjects are likely to include it in the answer. For example, if the premises are

some triangles are isosceles traingles
some triangles are scalene triangles

subjects may incorrectly continue on in the 'some' atmosphere and respond:

some isosceles triangles are scalene triangles.

A second theory offered for mistakes along these lines is the 'conversion' theory, according to which some premises tend to be converted to a different meaning. In particular, "All A are B" tends to be naturally converted to "All B are A", and an invalid conclusion may result. These conversions are very similar to the converse reasoning described above, where "If A then B" is converted to "If B then A"

Consistent with both the atmosphere theory and the conversion theory is the observation that the mathematical meaning of 'some' is not really the same as in real life. In real life, 'some' has a connotation of 'not all', while in mathematics it could include the possibility of all. Thus, the statement, "Some equilateral triangles are equiangular" is perfectly correct in mathematics, in the sense that there exists at least one equilateral triangle which is equiangular. But in real life, we would tend to feel that the statement hints that some equilateral triangles may not be equiangular.

An integration of the above ideas has been attempted by Erickson (1973), in a theory of representations. He claims that people naturally bring to mind something akin to Euler's circles when considering set inclusion. The statement, "All regular polygons are equiangular" is naturally represented in the mind as

a circular set of regular polygons contained in a larger circle of equiangular polygons. Errors arise when that representation shows the two sets as one and the same circle; in that case, the statement "All regular polygons are equiangular" is represented in the same way as "All equiangular polygons are regular", and so the two statements may be erroneously treated as equivalent. Thus, the representational theory explains the conversion effect described above, but does not explain *why* such an erroneous representation or conversion is made. To state that representations are made, but that sometimes they are the wrong representations, does not contribute very much to our understanding of the logic. The fact remains that the wording of the sentence, using various quantifiers such as 'not all' contributes to the difficulty of processing the sentence. This linguistic connection has not been sufficiently explored.

Logic in general

In this chapter various forms of logic have been discussed,and in each case theories have been cited which explain certain logical difficulties of students. In some cases, those theories even contradict each other; for example, theories of spatial imagery are used to explain the logic of comparisons in the three-series problem, in apparent opposition to more linguistically-oriented theories.

The problem with most theories is that they attempt to explain a phenomenon in terms of one variable only. They assume that there is one and only one way in which the brain processes information. But it is quite possible that different individuals process information in different ways. In fact, one individual may prefer different approaches depending on the task. For example, even though I may usually be a pictorially-oriented thinker, I may find a word-oriented approach to my liking in some instances.

Educational research methodology tends to reinforce this unitary approach. The typical experiment, wishing to compare method A with method B, may perform a *t*-test and find significant differences in favour of method A. The interpretation is usually that method A is therefore superior to method B. There is a hidden assumption that method A should now be used in preference to method B for *all* students. Actually, the experimental result may also be interpreted as meaning that a majority of students are more successful at method A. In this interpretation, there may be

many students who prefer method B. Moreover, we may ask whether students who have been exposed to *both* methods A and B are superior to those who have been exposed to only one method. This is a question which research has not adequately addressed.

For example, most teachers of logic prefer the Venn diagram/Euler circle method of representing syllogisms. Perhaps this is because mathematics teachers themselves are usually spatially oriented people to begin with, or perhaps their experience shows that this method generally produces better results in students than, say, a strictly verbal method. They may even realize that some of their students are not very spatially oriented, and may try to find time to tutor those students in other methods. But perhaps the best, and most overlooked, solution would be to expose all students to both methods, under the theory that the more ways of looking at an issue, the more complete will be the understanding.

In light of the experimental results, it is probably fair to state that students can attack syllogistic logic problems in various ways, such as by using spatial imagery, or by using verbal transformations. There is, however, a question prior to this one, namely, what makes children attack a problem at all?

Many experiments in logic, especially in non-Western countries, have found that subjects simply refuse to grasp the constraints of the problem at hand. They do not use the information given, and often bring in extraneous assumptions. They seem to miss the entire point of logical deduction, namely, that some statements follow necessarily from other statements.

This observation may be related to the point made throughout this book that formal mathematics and logic involves a departure from the context of the situation at hand. A person may be able to use perfectly sound logic in an everyday situation but may fail disastrously if the same problem is presented in context-free laboratory conditions. Roberge and Craven (1983) have attempted to examine logic in more context-dependent situations, and have obtained different results from the usual in-class logic tests. They conclude:

> These findings raise doubts about the usefulness of the context-free propositional reasoning tests administered in previous studies as predictors of student's performances in everyday reasoning situations. (p. 75)

This topic is discussed in more detail in Chapter 5, Problem Solving. For now, however, we leave teachers with the idea that when students are not performing well in deductive reasoning, teachers should ask whether the students are finding some difficulty with the problem itself, or whether they are failing to attack the problem at all.

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CHAPTER 5

PROBLEM SOLVING

The literature does not agree on precise definitions of the terms 'problem' and 'problem solving'. Some theorists may give the terms general meanings, in the sense that any situation which involves a choice in view of a goal is a problem, and a 'correct' choice (i.e., one which achieves the goal) is a solution.

Many journal articles use the terms 'problem' and 'problem solving' in connection with the 'word problems' found in most arithmetic textbooks. In this sense of the term, finding the answer to the question, "If John has 10 apples and gives 3 apples to Mary, how many apples does John have left?" is an example of the type of problem solving referred to in many articles. Since this perhaps restricted use of the term may be simpler to discuss, and since such a discussion may yield insight into more complex types of problem solving, we treat it first.

Word problems in arithmetic

One of the main difficulties of educational testing in general, and of research into problem solving in particular, is that thinking is done inside the brain and must be analyzed only from external indications of thought. It is possible to look at a child's answer to a problem, but does that answer provide any insight into the actual process by which that answer was obtained?

One solution, of course, is merely to ask the child to describe those mental processes. Psychologists term such a description a *protocol*. In general, however, articulate protocols are hard to obtain, especially from younger children. As we have discussed earlier, the focused articulation of one's thoughts is a higher order ability and is indeed one of the goals of mathematics education and of education in general.

Even if the protocol is rationally explained by the subject in an interview, it can only go so far in explaining how the problem was solved; there seem to be stages in problem solving

which cannot be tapped by introspection. It appears that key steps in problem solving must necessarily be accomplished subconsciously.

The protocol can at least point to the step(s) where difficulty was encountered. Children may be able to describe, for example, how they understand the problem, which operation they chose in order to solve it, and how they carried out the operation. The experimenter may then be more fully able to isolate where difficulties may have occurred.

To facilitate the protocol process further, experimenters may be able to ask probing questions along the way. In fact, a structured protocol is often used in which subjects are asked to describe only certain steps in the problem-solving process. The structure usually follows the order of steps normally used in problem solving; these steps have been analyzed and listed in some simple models by many mathematics educators. For example, Newman (1977) listed five steps in the problem-solving process:

reading
comprehension
transformation
processing skills
encoding

More precisely, students must first be able to read the problem. Second, they must then be able to understand what the problem is all about. Third, they must transform the words of the problem into mathematical symbols. Fourth, they must perform the required operation(s): addition, division, etc. Finally, they must be able to turn their numerical result into an answer to the given problem.

Using such a model, the experimenter can ask subjects a series of questions, such as, "Read the question aloud", "What is the problem asking you to do?", "Which operation(s) are appropriate", "Perform the operations: what is the final answer?" In this way it is possible to distinguish whether a student's difficulty lies in the reading, the understanding, or the transformation.

Newman was able to interview many low-achieving 7th grade students in order to determine the relative frequency of various types of errors. She placed 13% of the thousands of errors in the reading category, 22% in the comprehension category, and 12% in the transformation category. The first two

of these categories are definitely linguistically oriented, and the third category may be as well, as discussed below. Thus, at least 35% - perhaps as many as 47% - of the errors were linguistic in nature.

The experiment was repeated by Clements (1980) using 5th, 6th and 7th graders, with slightly different results:

		grade 5	*grade 6*	*grade 7*
% errors in category	reading	8	5	2
	compreh.	14	8	9
	transf.	27	25	27

Clements further analyzed the results for a group of average students and a group of low-achieving students in the 7th grade, and found:

	average	*low-achieving*
reading	3	3
compreh.	16	6
transf.	28	28

It appears, therefore, that errors made by low-achieving students are rather different from those of average students, and that students of average ability tend to make a higher percentage of their errors in comprehension. The ability to make the required transformation seems to be rather uniform across grade level and ability grouping. But the differences between Newman's results and those of those of Clements seem to suggest that the nature of the problems may influence the nature of the errors. In both cases, however, linguistic difficulties accounted for nearly half of the difficulties in problem solving.

What, then, are these linguistic difficulties? Are they syntactical (involving word order, verb tense, etc.) or semantic (involving meaning, vocabulary, etc.)? In order to answer this question, Linville (1976) devised problems which could be classified as having easy or hard syntax, and easy or hard vocabulary. This scheme led to four categories of problems: easy-easy, easy-hard, hard-easy, and hard-hard. When the problems were given to students and the results tested using ANOVA techniques, a significant F value ($p<.01$) was found, indicating that problems became harder as their syntax and vocabulary increased. Oddly, in looking for pairwise differences using a

Scheffe test, no significant pairwise differences were found. Thus, it was difficult to state whether the syntax alone or the vocabulary alone contributed to the problem-solving difficulty, nor could it be concluded that the combination of syntax and vocabulary was a stronger deterrent than either one in isolation. All that could be said was that the linguistic variables somehow contributed to difficulty in problem solving.

Linguistic cues

If the wording of a problem affects children's strategy to solve a problem, one may search for the linguistic cues which prompt them to opt for one strategy or another, and one may ask whether those cues are the *only* factors which affect the choice of strategy.

For example, addition problems often contain words and phrases like 'total', 'in all', or 'altogether'. To what extent do children focus on these words and, perhaps without even reading or understanding the rest of the problem, decide that addition is the required operation? The experimental evidence suggests that such cues can play some role in the choice of operation.

Carraher (1988) hypothesized that young children view addition as a transformation in which quantities increase, and therefore any wording in a problem which tends to suggest that numbers are getting bigger tips the child to use addition. Likewise, subtraction involves decreasing numbers, and so words relating to decreases are cues that subtraction is to be used. Similarly, problems involving rates (e.g. "each apple costs $3") usually require multiplication, while problems involving sharing (e.g. "how many does each get?") tend to require division, and so one would expect children to apply multiplication to any problem using rates, etc. In order to test these hypotheses, Carraher asked children to solve addition and subtraction problems in which the amount sometimes increased, sometimes decreased, along with multiplication and division problems involving rates and sharing. Table 5.1 illustrates the choice of operations used by children at various levels:

Table 5.1
Percentages of correct choice of operation by problem type and by level of schooling

Problem type	*1st grade*	*3d grade*	*5th grade*
Amount increases; addition required	93	87	93
Amount decreases; addition required	68	43	70
Amount decreases; subtraction required	81	97	93
Amount increases; addition required	16	47	31
Rate situation; multiplication required	33	100	70
Sharing situation; multiplication required	22	90	70
Sharing situation; division requir	45	100	90
multiplication required	12	44	20

The results clearly confirm the hypothesis that children expect to use certain operations in certain types of problems. But it is possible to observe changes from grade level to grade level. Apparently, problem-solving strategies change, but not necessarily for the better.

Nesher, *et al.*(1982), have studied these effects in detail, and have elaborated a hierarchy of fourteen semantic categories in which problems are classified by type ('combine 1-2, change 1-6, compare 1-6') and by four developmental levels. As an example of type, 'combine' problems ask questions about the union of two sets, or perhaps about one of two disjoint subsets, as in the problem "There are 3 boys and 4 girls. How many children are there altogether?" As an example of developmental levels, at level

4 operations become reversible, and so it is only at this level that "the child is able to coordinate directional terms without ignoring them, by employing the reversed relation description which is needed for choosing the correct operation. Thus he is able to read the word 'more' in the text, and yet perform a subtraction operation." Nesher concluded by identifying at which developmental level each of the 14 types of problems can be solved by children. Such a scheme should be of use to teachers, for they can be aware what types of word problems are likely to be difficult for students at various levels, and can therefore plan their instructional sequencing accordingly.

The evidence just presented shows that isolated words such as 'more' are used exclusively as cues to a greater degree at lower levels of understanding. Younger children may have few problem-solving tools in their mental bag-of-tricks, and so the verbal cue 'more' may be the only one they recognize. At older levels, they have mastered other tools, and when scanning their array of strategies in response to a word problem, they may decide that this particular verbal cue is not the relevant one.

In order for students to select the correct problem-solving technique, they must understand the problem in its entirety. Teachers must insist that students read an entire word problem before solving it. There is evidence (e.g., Hinsley, 1977) that students form judgments early on in the reading of problems. For example, if the problem starts, "In a sports car race....", students may snap to the judgment that the question is a distance/rate problem and will begin planning solution strategies accordingly.

It is usually the lower achievers who resort to the strategy of verbal cues, and they often do so incorrectly. This suggests that, unlike the better students who have grasped what the problem is all about, the low achievers are grasping at straws in an attempt to find some short-cut to solving the problem. It also suggests that the method of verbal cues is not very appropriate. Teachers may believe they are doing children a favour by showing them these verbal tricks, but in so emphasizing the superficial aspects of the problem, they may be leading children away from some of the primary objectives of the mathematics lesson, namely, the true understanding of problem statements and the solving of problems from understanding rather than from superficial trickery.

Scanning strategies

There is an almost tacitly accepted assumption that problem solving involves a mental search or scanning of possible strategies. As discussed above, young children may have a very limited collection of strategies. For example, they may not even have mastered all of the basic arithmetic operations, and their linguistic development may not allow them to pick up some of the subtleties of word problem statements. They may focus on certain elements of the problem statement's surface structure, such as the word 'more', without understanding its deeper structure.

But even if a child has acquired a large number of strategies, how does the scanning process work, and how does the brain select which strategies are appropriate or inappropriate? Part of the answer must lie in the nature of the 'cognitive structures' in the brain. We have seen in Chapter 3 how concepts may be formed hierarchically, and that the existence of higher order concepts, i.e. categories of categories, enables the mind to move from one branch of the hierarchy to another. The same process must be at work in problem solving, as the brain tries to find similarities between the problem at hand and problems experienced in the past. Surely the similarity of some words is one easy link: if in the past a problem with the word 'more' required an additive solution, then perhaps the problem at hand, which contains 'more' will also require the same sort of solution. A rich set of several comparisons may convince the solver that the present problem resembles a previously encountered one in so many ways that the solutions must be similar.

Thus, a problem may resemble a previously encountered one in a direct linguistic way, but there may be similarities in form, content, or a host of other aspects, depending on the way the problem is represented by the mind. This representation may be purely linguistic - the solver simply remembers the wording of the problem - but it may also be schematic or spatial. Davis (1985) listed six categories of representations:

1. natural language statements (e.g.,"sulphates are soluble, sulphides are not");
2. representations that permit logical inference making
3. storage of example or instances;

4. constructing new representations by building upon existing representations;
5. representations of algorithms, procedures, etc.; and
6. representations of formulas.

With reference to the first of these categories, Davis remarked,

> Natural language statements probably play a much smaller role than is often imagined. Of course, natural language and logical inference are *the* key processes in "finished" mathematical discourse of certain types, but we have not found them figuring all that prominently within the process of human *thinking*, even where mathematical tasks are involved. (pp. 86-87)

Even so, the decision to use one strategy over another may be based on a combination of representations. A given problem may be represented similarly to several others, and similarities may be noticed in many aspects; but it may be that if a linguistic similarity is coupled with a similarity in formula, for example, the reinforcement of the two similarities may cause the solver to choose that connection over another one. It is the similarity on multiple levels which appears to be a key factor in realizing that a solution has probably been found.

This is the view of other researchers, for example, Lemoyne and Tremblay (1986):

> In interpreting concrete mathematical problems, subjects associate problem-solving processes already stored in their memory with certain contextual, relational, or lexical data, or with all three of these categories of information, good problem-solvers being distinguished from the rest by the richness and precision of their associations. (p. 101)

They go on to explore the relationship of language to the process, and realize that, as discussed in Chapter 3, that language is inextricably bound up with concepts of all sorts, not just the purely lexical ones. It follows that:

> The development of such an heuristic method may perhaps only be achieved if the subject already

possesses the linguistic skills enabling him to recognize certain formulations of a similar situation.

It seems fair to say, then, that it is this central transformational step which is crucial to problem-solving activity, not only for the usual type of arithmetic word problems, but for problem solving in general. It is time we turned our attention to the more general case.

General problem solving

Educators have been thinking for some time about how people solve problems in general. There seems to be an underlying assumption that there is a basic problem-solving mechanism in human beings, so basic and so innate that it is virtually impossible to define. The well-known educator John Dewey investigated the problem as early as 1910, in his book, *How We Think*, in which he preferred the term 'reflective thinking', to illustrate that this type of thinking is goal-oriented. He used this term to refer to any situation which causes some doubt or uncertainty in our minds, causing a choice to be made. Even the decision of choosing the branch of a fork in a road involves reflective thinking, provided we know where we want to go (the goal).

Dewey divided the problem-solving process into five stages, which may serve as well as any for the purposes of our analysis. These are: (1) the awareness of uncertainty; (2) formulation or articulation of the problem; (3) searching for possible strategies from past experience of similar problems; (4) deductions from the hypotheses; and (5) verification. These stages correspond roughly to the stages of solving arithmetic word problems as described above. The most vague of the stages is the fourth one, which probably corresponds to the crucial transformational stage already described.

In fact, some authors describe problem solving only in terms of the transformational step, or at any rate, they emphasize this step practically to the exclusion of the others. They no doubt feel that stages one, two, and perhaps even three are self-explanatory: *of course* the solver must be aware of the problem and must be able to know the constraints of the problem. For example, Wickelgren (1938) described his general problem-solving method:

> Draw inferences from explicitly and implicitly presented information that satisfy one or both of the following two criteria: (a) the inferences have frequently been made in the past from the same type of information; (b) the inferences are concerned with properties that appear in the goal, the givens, or the inferences from the goal and the givens. Very few steps are required to solve the [insight] problem. What is necessary is to make the one critical transformation of the givens that essentially solves the problem. (p. 23)

One of the most influential contributors to the theory of problem solving was Max Wertheimer (1961). Wertheimer was relatively unconcerned with the statement of the problem. The only thing which mattered was the transformational step, with some reference to the search step. Wertheimer was the primary exponent of the *gestalt* school of thought, which viewed 'productive thinking' as occurring in the context of broad structures. It is not possible to repeat that entire theory here, but certain features may be cited. Central to the theory is the notion that "the situation calls not merely for piecemeal factual truth, it calls for 'structural truth'," and that truth is obtained by "the operations of grouping, of centering, of reorganization, etc. adequate to the structure of the situation."

This idea of structure pervades all of Wertheimer's work. The mind works within the framework of the gestalt, but the problem is solved by looking at that gestalt in such a way that the solution to the problem becomes apparent:

> Thinking consists in envisaging, realizing structural features and structural requirements; thereby changing the situation in the direction of structural improvements, which involves....that inner structural relations - fitting or not fitting - be sought among such disturbances and the given situation as a whole and among its various parts; realizing structural transposability, structural hierarchy, and separating structurally peripheral from fundamental features. (p. 235)

Wertheimer is silent on the effect of language on the process of transforming structures. Not only does his entire major opus, *Productive Thinking,* never refer to language, one of its major experiments is done with deaf-mute subjects, many of whom are quite capable of solving the problems presented. We may take this as evidence that Wertheimer is assigning to language a very minimal role indeed in the problem-solving process.

Likewise, Polya (1947), another investigator of problem-solving activity and author of the well-known *How to Solve It,* has very little to say on the role of language in problem solving. His only comments are on the role of the teacher's presentation of the problem and hints about its solution to the student. The key objective of the teacher is to get the student to focus on what is required and to compare the problem under consideration with other previously solved problems in the student's experience. The teacher can be useful in reminding students of the realities of the problem: "Did you use all the data? Do you know a related problem? What is the unknown?" Otherwise, Polya does not appear to attach much importance to the language actually used by the problem solver. Like Wertheimer, Polya sees the role of language primarily as useful into coming to grips with the problem rather than with the actual leap of finding its solution. Once the solver grasps the problem and enters into its framework, language is relatively unimportant in finding its solution.

In the above discussion on arithmetic problem solving by children, it was observed that the wording of problems could have an effect on students' ability to solve problems. One way of reconciling this fact with Wertheimer's ideas is to note that Wertheimer was concerned only with the transformational step. We might say that language has a natural effect on how the subject understands the problem, but once that understanding has been established, the crucial step is accomplished largely independently of language.

Research supports the claim that deaf-mute subjects can solve many problems as well as hearing subjects. Furth (1966) argued further that problems can be solved in the absence of any form of symbolic system. and that intelligent thinking "does not need the support of a symbolic system, as it exists in the living language of society." Zepp (1986) also found that a deaf-mute who had never been to a school or even a town, and hence could not be said to have been exposed to a symbolic system, was still able

to solve the 'four-card' problem described in Chapter 5. Of course, some problems are more language-oriented than others, and will naturally require linguistic ability. All that is being discussed here is whether there is any problem solving *at all* that can occur without language, to which the answer is a strong affirmative. It will remain for future research to determine what sorts of problems are related to language, and to what extent.

If Wertheimer and the gestaltists can be criticized, it is on the grounds that, despite their emphasis on mental structures, they do not describe where those structures come from. Piaget does a far better job of doing this, as we have seen. We have learned from him and others such as Bruner that language is inseparably tied to the formation of concepts and mental structures. Therefore, the type of language used by a problem-solver may affect the type of mental structure used in solving the problem, and hence the nature of the solution.

Evidence to this effect exists. In an experiment by Glucksberg and Danks (1975), the problem was to complete an electric circuit when the wires were not quite long enough. Several items were available to aid in the solution, one of which was a screwdriver. One solution would be to use the screwdriver's metal portion to conduct the electricity between the ends of the wires. But subjects tended to overlook this solution because screwdrivers are habitually used only for turning screws, and therefore the 'functional fixedness' of the screwdriver excluded it from being used in a novel way in solving the problem. In fact, if subjects are asked to turn screws directly prior to the experiment, their tendency to ignore the novel use of the screwdriver increases. This is a good example of the effect of the context or environment on the ability to solve problems.

The relation to language is that the name of the item may affect functional fixedness. For example, the English word 'screwdriver' reinforces its function in driving screws. If the word in another language meant only 'metal rod', subjects who spoke that language might be far more open to the solution of allowing the 'metal rod' to conduct electricity from one wire to another.

In Chapter 7, discussing the Whorf hypothesis, we saw that verbal descriptions could influence subjects' perception of problems. For example, the result of Carmichael, *et al.* (1932) that the drawing of two circles joined by a line was remembered

differently when described as 'spectacles' from when it was described as a 'dumbell'.

The givens of any problem may be stated explicitly or implicitly. In the implicit case, the problem solver must extract some extra information from the givens, and the type of language used may affect the facility with which the solver can extract the implicit information. Certain languages force the speaker to be aware of certain aspects of the situation, such as gender. For example, consider the following well-known problem:

> A father and his son have a car accident in which the father is killed and the son is critically injured. The ambulance arrives at the hospital and the child is taken to the emergency room. The operating doctor takes one look at the boy and states, "I cannot operate on this boy because he is my son." Can you explain the situation?

The answer is, of course, that the doctor is the boy's mother. But in cultures where doctors are usually men, people often have trouble solving the problem. If the word 'surgeon' is used, the difficulty is even magnified. In a language where gender must be supplied for every noun, this problem would not even be considered a problem, because the crucial information would be made explicit in stating the gender of the doctor. There would be nothing to explain. Even in English, if the word 'nurse' had been used instead of 'doctor', there would be nothing to explain.

The difficulty of most 'insight' problems lies in the implicit information in the givens, and it is the task of the solver to extract that implicit information from the explicitly stated text of the problem, usually by focusing on some novel aspect of the givens. Any language causes its speakers to emphasize some aspects of situations at the expense of others. The word order of a sentence, the vocabulary, and even the intonation contribute to the focus on some aspects of the problem.

By the same token, language may obscure certain details by causing the solver to focus on precisely the aspects which are most inappropriate to the solution. The 'functional fixedness' of the screwdriver problem is a case in point, in which the English word 'screwdriver' forces the solver to focus on the driving of screws rather than on the conducting properties of the metal. The doctor problem might be made even more difficult if the word for 'doctor' had an explicitly masculine aspect, 'medicine man', for

example, which might force the solver to view the doctor as a man.

In fact, any way in which language relates to the context of the problem-solving environment can affect the solution to the problem. It is basically the effect of the context which Wertheimer fails to discuss. The wording of a problem can clearly provide clues for the search activities, and the search activities themselves may involve sets of similar words or sounds. The language of the problem may even suggest the type of transformation of the structure or way of looking at the data which is necessary to solve the problem, perhaps by stating precisely just those aspects of the given data which must be focused on in order to transform the problem appropriately.

Information processing approach

In this age of computers, it is perhaps inevitable that the human mind should be compared to a computer, and that human thought should be treated as information processing. Let us assume for the moment, then, that the human brain is nothing but a computer, and let us ask how we can know anything of its internal workings. Simply observing responses (outputs) to stimuli does not guarantee an answer to what actually goes on inside the brain. Two people may use radically different mental processes to obtain the same solution to a problem. It is possible, however, to measure such data as response times, under the assumption that if it takes longer to respond to one stimulus than to another, then the mental process must be more complex. We have seen experiments of this type when discussing right-brain left-brain differences: a shorter response time to an item shown to the right visual field is taken as evidence that the item is being processed by the left hemisphere.

The notion of a search operation for finding strategies to already-experienced problems similar to the problem at hand sounds very much like a computer operation. But there appears to be something more than a routine search operation, because a computer search may be a random one, while a human problem-solving search has a purpose and is guided by that purpose. The search is constructive. There is the added element of a deeper understanding of the problem. One exponent of the information-processing school is Greeno (1977), who proposes that the study

of understanding a problem is similar to the study of understanding language.

For one thing, the goal of the problem must be kept constantly in mind, just as the goal of speech (namely, the conveyance of a message) must be kept in mind. We are all familiar with the phenomenon of losing our train of thought, that is, losing sight of the message in mid-sentence and having to abort the sentence. Similarly, students have difficulty memorizing proofs to theorems if they do not understand what the proof is about. Greeno states:

> Often, understanding in problem solving depends on some general idea that provides a kind of theme or gist of the solution, just as understanding of a written or spoken message often depends on a global theme that may be provided as contextual information. (p. 255)

This description sounds very gestaltist, with the 'theme or gist' corresponding to the gestalt.

In this vein, one of the early researchers in the field, Duncker (1945) proposed that mathematics teachers should use organic proofs which always move towards the goal of the proof, rather than mechanical proofs, where "each step is merely justified on the basis of material derived earlier without relating it forward to the goal of the problem." If a student is having difficulty with a proof, perhaps the best question a teacher can ask is, "What are you trying to prove?"

It is this goal orientation which almost all theorists hold as the *sine qua non* of problem solving, and which is perhaps the most difficult aspect of artificial intelligence of computer problem solving. In relation to our discussion of sociolinguistics in the classroom, we have seen that lower-class children are not nearly as goal oriented as middle-class children. It is natural to suspect that lower-class children are not nearly as good problem solvers as middle-class children. Research evidence bears this out.

Problems in context

While the immediate context can affect problem-solving activity, the key to productive thinking is to view the problem from several viewpoints, including ones which one is not

accustomed to. If some children are more or less glued to one viewpoint, namely that of the present context, it may be difficult for them to transform a problem into a model that is viewed from the single viewpoint which makes it clear how to solve it.

This line of reasoning leads to a discussion of the artificial problems in mathematics textbooks and in laboratory situations. For children in non-Western countries, such artificial problems, which may begin, "suppose you have ten apples", for example, are contradicting the context of the real situation, for the child patently does not have ten apples. Thus the problem is taken out of the context of the real situation, and the children may find it impossible to enter into the domain of the hypothetical problem. Many studies have examined the cognitive abilities of natives in remote or 'primitive' places. In order to be considered seriously by academic journals, the authors must attempt to set up laboratory conditions which hold as many variables as possible constant. In these highly artificial conditions, the native subjects usually perform very badly by Western standards. The authors may naturally conclude that these un-Westernized peoples are poor problem solvers, or that they are are deficient in some basic skills.

Cole and Scribner(1974) discuss this problem lucidly. They point out, for example, that desert dwellers like the Kalahari Bushmen or the Australian Aborigines, whose lives may depend on the amount of water available, perform badly on Piagetian tasks involving the conservation of water. They ask:

> Can we imagine an adult [Aborigine] who would pour water from a small bucket into a larger one and believe that the amount of water has been decreased by this act? In desert communities where water is a treasured commodity, everyone can be expected to conform to certain laws of conservation. (p. 152)

Surely, in a context of life-and-death reality, the Aborigines would conserve volume. We must accept that they perform differently in artificially created experimental conditions.

The most common failure of unschooled subjects in problem-solving tests is the inability to accept the hypothetical 'set' of the problem. They often ignore critical data, and they often use data not given in the problem statement. They appear to answer questions intuitively from the totality of their prior experience, and have no notion that the problem statement

establishes an isolated setting within which the solution can be found. Their response to hypothetical problems is often along the lines of "How should I know, I wasn't there." Similar findings have been reported from all over the world, so there seems to be something universal about this type of response (see, e.g., Luria, 1971).

For the purposes of this discussion, we must ask whether this 'failure to accept the logical task' (the phrase was coined by Henle, 1962) is of linguistic origin. Evidence is sparse indeed, but it is possible to cite data that suggest subjects are not grasping the meaning of the words of the problem. They often cannot repeat the problem once it has been stated, and frequently even after it has been repeated. The following is an excerpt from a problem-solving session reported by Cole and Scribner (*op. cit.*) from Liberia:

> *Problem*: Some kwi (Western) people are wealthy. All wealthy people are powerful. Are some kwi people powerful?
> *Subject's answer and reason:* Yes, it is because some kwi are wealthy and they have power.
> *Subject's first recall [of the problem statement]*: Some kwi in this town are wealthy. They have power.
> *Experimenter*: What question did I ask you?
> *Subject:* Do some kwi have power?
> Experimenter reads the problem again.
> *Subject's second recall*: Some kwi are wealthy. They have power. Do many wealthy men have power?

On the first recall, the subject misstated the general premise, "All wealthy people are powerful", but correctly repeated the question. On the second recall, he again missed the general premise, and misquoted the question as well. We cannot merely ascribe these mistakes to faulty memory, for the ability to recite long folk epics is well-known in non-Western cultures. Rather, we sense that the subject has not grasped the problem. He has understood the words - the surface structure, if you will - but the mind has not accepted the gestalt of the problem, for lack of familiarity with this kind of I.Q. test item.

This is an area where very little research has been done, and which promises to be very fruitful. If we could put our finger on what aspects of problem statement or even problem context

affect the solver's acceptance of the problem as a whole, we might reach many underachievers.

It seems clear that the wording of problems can affect their difficulty. The dilemma for the teacher is how far to go in making the problem easier through its wording. If the wording is such that the solution is a giveaway, it is no longer a case of problem solving. But if the wording is so abstruse that the problem becomes virtually impossible, then no problem solving takes place in this case, either. It is the job of the teacher to find that happy medium which enables the students to enter into the framework of the problem but does not give away the solution. The teacher may have to explain the wording of exercises in the textbook, or otherwise reword the problem.

Conclusions

There is clear evidence that 'structure' is the key word to solving problems. The solver must understand the structure or gestalt of the problem, and must relate that structure to previous existing structures within the mind. The major question for teachers, then, is how to aid in the development of cognitive structures. Chapter 3 discussed ways in which hierarchies of concepts could be fostered which facilitate the search process. The same remarks may be made here: multi-level hierarchies consisting of layers of concepts seem to provide richer insights into problems. The generality and richness of language used by the teacher can help to foster these types of structures.

The important point is that the solver must be conversant with several different strategies and must be able to consider the givens of a problem from several points of view. This is the aspect of problem solving where unschooled students or lower-class children experience difficulty. These groups' cognitive styles root them relatively fixedly to the context of the here and now. They have difficulty visualizing an artificial problem with hypotheses contrary to what they know to be the case in real life.

Bernstein's followers may have some light to shed on this question. They would undoubtedly claim that the context-dependent language of lower-class children makes it very difficult for them to examine the hypotheses of a problem independently from the context of the situation. It is particularly for these children that teachers should act out problems concretely or

otherwise pose real problems which children have an interest in solving.

But the main message of this chapter for teachers is that the more varied their children's experience, the better problem solvers they will become. We may draw a parallel with results of experiments on bilingual education, as seen in Chapter 8. Children who have crossed a certain threshold of competence in their second language appear to have a cognitive advantage over monolinguals, since they have the advantage of viewing a problem through two systems of thinking. On the other hand, if the second language has been partially learned in such a way that the competence threshold has not been attained, thinking may be confused even in the first language, and negative interference may occur. Likewise, children should be made truly competent in several problem-solving strategies, so that the child is confident of being able to apply either to a given problem. If one of the strategies is not well understood, the whole process may be confused in the child's mind, and the negative interference of the badly understood method may cause the child to fail to solve the problem even by the first method.

Thus one of the contributing factors to problem-solving ability may be the child's exposure to new experiences, so that a variety of possible strategies may be at the solver's fingertips. It is the role of schools to provide some of those new experiences, in particular the opportunity to see the world through different eyes. A person who has experienced problems from many points of view is more likely to select the correct strategy for solving those problems.

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CHAPTER 6

CORRELATIONAL STUDIES

This chapter is devoted to the relationships between skills and achievement in mathematics and in language. In particular, the question of correlations between various mathematical test scores and those on language tests are investigated. In order to do so, it is necessary to deal with some preliminary questions, namely:

1. Is ability in mathematics a single skill, or is it rather a composite of several separate and independently measurable skills? For example, is ability in geometric/spatial concepts strongly related to ability in arithmetic/numerical concepts? Is problem solving related to manipulation of numbers via the four operations?

2. Is language proficiency a single skill, or rather is it a composite of, say, reading skill, writing skill, oral communication, and listening ability? High correlations among these skills would be an indication that all are simply manifestations of a single skill in overall verbal ability, while low correlations would indicate that they are independent skills.

It is also necessary to make the usual disclaimer concerning correlation and causation. Even if high correlations were reported between such skills as mathematical problem-solving ability and reading ability, we should not jump to the conclusion that a student's mathematical problem solving can be directly improved by holding extra reading classes.

On the other hand, if experimental data convinced us that no correlation can exist between the two variables, then there would be *a forteriori* no causal relationship. In this case, we could conclude that no amount of improvement in one skill will lead to an improvement in the other skill.

Is mathematical ability a single skill?

Mathematics educators may be familiar with a diagram attributed to Guilford (1959), shown in Figure 6.1.

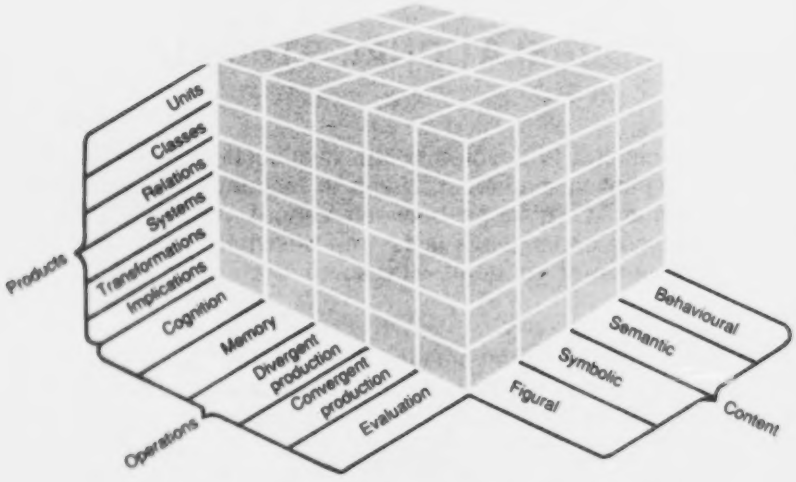

Figure 6.1 Three-way structure of cognitive skills.

The Guilford schema represents the view that abilities can be divided into separate and independently measurable abilities. A second model, proposed by Vernon (1960), shows abilities as branching from a more general ability in a hierarchical way.

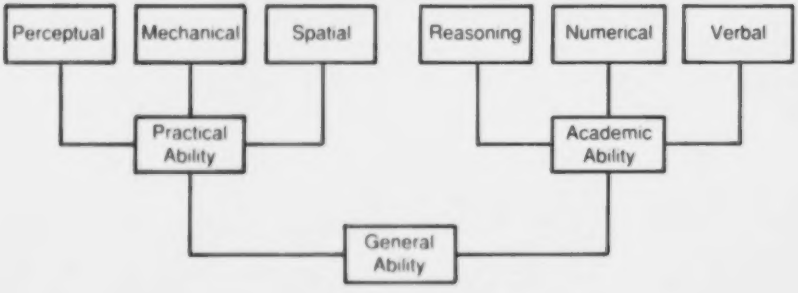

Figure 6.2 Branched hierarchy of abilities.

In order to break general aptitude into its component factors, the technique of factor analysis is often used. Various subtests or even individual test items are grouped into independent factors. For example, there may be four subtests, say tests A, B, C and D. The factor analysis may find two major factors, each of which is a linear combination of scores A, B, C and D. For example, factor one may be .54A + .33B + .12C + .02D, and it may be found that test score A correlates with the general factor at .76. The difficult task for the theorist is to assign names to the various factors, names which reflect the composition of the factors.

As a concrete example, the data in Table 6.1 describe a factor analysis of six selected United States Air Force aptitude tests (Thorndike & Hagen, 1977), showing the correlations of the six tests with the three principal factors.

Table 6.1 Factor analysis of six aptitude tests

	Factor		
	1	2	3
Reading Comprehension	**.72**	.04	.04
Navigator Information	**.70**	.14	.08
Numerical Operations	.08	**.71**	.00
Dial and Table Reading	.28	**.57**	.30
Speed of Identification	.13	.37	**.58**
Spatial Orientation	.13	.02	**.63**

Clearly, factor one is composed primarily of reading comprehension and navigator information, and might be given a title something like 'information comprehension'. Factor two might be titled 'numbers and tables' due to the high loadings of tests three and four.

Two points must be kept in mind when reading reports of results of factor analyses. Firstly, the names of the factors are the creations of the analyst. For example, if a journal article reports that one of the factors is a numerical ability factor, that name is the author's choice, and another author may give it a different title. In this way, authors can greatly bias the reporting of factor analysis by assigning factor names which reflect the authors' preset ideas.

Secondly, the loading of two subtests onto the same factor does not necessarily imply that they are identical, or are even highly correlated. In the above example, reading comprehension and navigator information correlated with factor one at levels .72 and .70, respectively, as shown in Table 6.1. But the correlation of reading comprehension with navigator information was measured at only .50. Similarly, numerical operations and dial and table reading loaded onto factor two at .71 and .57 respectively, but correlated only .44 with each other.

In a study by Roberge and Flexer (1981), junior high school students were given a battery of eight tests on various skills, including mathematics computation, mathematics concepts, and mathematics problem solving. All three of these aptitudes loaded onto a general factor, with loadings of .84, .87, and .89, while the intercorrelations were as follows:

	computation	concepts
concepts	.80	
problem solving	.81	.80

These data indicate a strong unitary ability in mathematics which underlies the three areas of computation, concepts and problem solving.

Another well-known and well-tested aptitude test used in educational testing is the General Aptitude Test Battery (GATB). It is divided into several subtests, which are then subjected to a factor analysis and broken into nine factors, which include verbal aptitude, numerical ability, spatial aptitude, and form perception. For now, the discussion can be limited to the three mathematically based abilities, namely numerical ability, spatial aptitude and form perception. Typical intercorrelations, in this case for final year secondary school students, have been reported as follows (Thorndike and Hagen, 1977):

	Numerical	Spatial
Numerical		
Spatial	.34	
Form Perception	.42	.48

These correlations, while significantly different from zero, are not very high. This indicates that while there is considerable overlap among these three skills, they are far from identical.

Similar intercorrelations have been measured on other aptitude tests such as the Differential Aptitude Test (DAT) and the Employee Aptitude Survey. Most of these tests are similar in their choice of subfactors: numerical ability, spatial ability and sometimes abstract reasoning ability. It may appear, therefore, that the tests have a built-in bias towards breaking mathematical ability into certain subcategories: if the selection of categories is done *a priori*, then of course mathematical ability will not be seen in the light of any other categorization scheme. However, the selection is not really an *a priori* one. Test items are subjected to factor analysis and those test items which load onto a common factor are given a certain name. For example, two different subtests of the GATB, one in computational ability, the other in arithmetic reasoning (i.e., verbally stated quantitative problems), both load highly onto the same factor, which is then called numerical ability. Thus, analysis of the test results seems to show that computational ability and the ability to solve word problems are highly correlated and can be treated largely as a single skill. But in light of the second consideration discussed above, it may still be the case that computation and word-problem solving are not highly correlated with each other.

The results above are associated with secondary school students. Results are slightly different for younger children. One series of tests for young children is the set of McCarthy Scales of Children's Abilities. Original testing in the U.S.A. led to a factor analysis in which five factors were identified, labelled Verbal, Perceptual-performance, Quantitative, Memory and Motor. Subsequent testing in the U.K. (Kaufman, 1975) with children of ages 2 to 8 did not support the choice of these categories. There, the three quantitative tests - number questions, numerical memory, and counting and sorting - did not load highly onto a single factor. The counting and sorting test, for instance, loaded partially onto factor 2, which was labelled 'verbal/quantitative' and partially onto a factor 3 which included opposite analogies, verbal fluency, and conceptual grouping. As a consequence, the author concluded, "The quantitative scale should be interpreted cautiously with children under 5 years, as it may not be distinct for the other scales, particularly the verbal scale."

A more recent replication with British 4-year-olds (Trueman *et al.*, 1984) confirmed support for a general cognitive

factor, a verbal factor, a motor factor, but *not* for a quantitative factor. Thus, at least for younger age groups, we should not speak of distinctly quantitative skills as independent from verbal or other skills.

So far, only the results of factor analyses have been discussed here. What about direct correlations between various mathematical tests, for example numerical concepts and geometric concepts? One study by Choat (1977) investigated the acquisition of these skills by young children. Six-year-olds were tested in each of three successive terms. The correlations between arithmetic and geometry scores were .69, .79 and .84 in terms I, II and III, respectively. The sample was also blocked into groups of very good, good, average and poor language ability, and the intercorrelations were as follows:

Table 6.2 Number/Geometry by Language Ability Rating

	Very Good	Good	Average	Poor
Term I	.533	.784	.625	.677
Term II	.571	.841	.771	.650
Term III	.597	.876	.878	.580

The correlations are quite high, indicating that the skills required for acquiring numerical concepts in school are roughly the same as for geometrical concepts. This is particularly true for students in the good to average range on language ability. One explanation for the high correlations is that both number scores and geometry scores are measures of concept formation, and thus are largely measures of a single skill.

Many studies have been done to measure correlations between various mathematical abilities, and most arrive at similar conclusions: the correlations are significant but sometimes not very high, ranging between .35 and .80. Such correlations include numerical estimation vs. trial-and-error problem solving (Rodney, 1971), paper-and-pencil problem solving vs. 'thinking-aloud' problem solving (Zalewski, 1974), as well as pairwise correlations among "number and operations; geometry and measurement; relations, functions, and graphs; logical thinking; mathematical sentences; and applications." (Zoet, 1973). Among university students such correlational studies often fail to find significant relationships. For example, Zepp

(1982b) found very low correlations among Lesotho university students between skills in logical concept formation and in logical concept application. We may therefore conclude that mathematical skills have common elements, perhaps even one common aptitude called 'mathematical ability', but that common ground is far from sufficient in explaining all the variability in scores on various mathematical tests.

Is language proficiency a single skill?

The question of language proficiency has generated more debate than that of mathematical proficiency. There have been far more studies using correlational analysis and factor analysis to investigate the nature of linguistic competence. This is probably because the issue of linguistic competence is at the very heart of theoretical discussions over the nature of language and consequently the nature of the human mind. For example, a correlation between speaking ability and listening ability might shed light on the nature of the verbal communicative process.

As in the case of mathematical aptitudes, there is no clear-cut answer, and there is in fact considerable debate over whether language ability consists of one or many skills. At one extreme is the model of Hernandez-Chavez, *et al.* (1978), which divides language proficiency into 64 independent skills. Unfortunately, experimental evidence does not support the existence of such a wide range of aptitudes: most correlational studies find high intercorrelations among many of the posited aptitudes, indicating a high degree of interdependence among them.

Perhaps the strongest support for the multi-dimensional model comes from neurophysiology. Lesions in different parts of the brain may cause different linguistic deficiencies. For neurophysiological evidence, one naturally turns to the work of Luria:

> Disturbance of this "speech zone" can lead to the development of aphasia. These disorders, however, are not identical, and lesions of different parts of this zone, directly affecting one or other aspect of speech, cause structurally different forms of aphasia. (Luria, 1963, p. 135)

> Two aspects of language - unseparable in normal speech - can be separated by brain pathology. (Luria, 1974, p. 12)

At the other extreme is the theory that language proficiency is basically one skill. A major proponent of this theory is Oller (1976), who provides considerable statistical evidence to support his claims. His main source of evidence is the factor analysis of many tests of language proficiencies of several types. The results of such analysis usually produce a single factor which accounts for a large amount of the variance in scores, and onto which most of the 12 individual tests scores load very strongly. Table 6.3 illustrates the loadings:

Table 6.3 A principal components analysis over 12 English language proficiency tests ($N = 573$)

Tests	loading on *g*
Imperatives	.89
Paraphrase recognition	.87
Repetition	.89
Oral question and answer	.79
Multiple choice reading	.92
Multiple choice cloze	.93
Written questionnaire	.86
Standard cloze	.90
Expert rating of speaking	.90
Self rating of speaking	.83
Self rating of reading	.81
Self rating of writing	.79

The *g* factor clearly accounts for a large amount of the variance on all 12 tests. These tests include tests of reading, writing, speaking, listening, in short, a wide variety of linguistic skills, and the results appear to confirm Oller's thesis that all of them are basically a single linguistic proficiency. He goes on to cite other studies done with more restricted groups of students, for example students whose first language is not English. In a study by Caufield and Smith (1981), Spanish-speaking secondary students were given a battery of seven tests of English (their second language),

constructed by the Modern Language Association (MLA). The factor analysis produced loadings onto a single factor, as shown in Table 6.4.

Table 6.4 A principle components analysis over seven foreign language proficiency tests for a sample of subjects with a restricted range of proficiencies (N = 28)

Tests	loadings on *g*
Cloze test	.92
Noise test	.94
MLA listening	.81
MLA speaking	.90
MLA reading	.90
MLA writing	.95
Oral interview	.86

Once again, the high loadings on a single factor are an indication of a unitary linguistic ability which encompasses all the major areas of language use: reading, writing, speaking, and listening. A cloze test, incidentally, while familiar to language teachers, may not be so familiar to mathematics teachers. It is described in the *Longman Dictionary of Applied Linguistics* (Richards *et al.*, 1985), as follows:

> In a cloze test, words are removed from a reading passage at regular intervals, leaving blanks. For example every fifth word may be removed. The reader must then read the passage and try to guess the missing words.

In fact, Oller suggests that if tests of the various linguistic skills measure essentially the same thing, then only one test should suffice for all of language testing, and the especially high correlations of cloze test results with the general language factor suggest the cloze test as a good overall test of language proficiency.

In spite of such strong evidence as that presented above, many theorists have adopted a theory in which language proficiency is basically not one skill, but two. This is the view of Cummins (1986), who posits a distinction between basic interpersonal communicative skills (BICS) and cognitive/

academic language proficiency (CALP). This distinction helps to clarify, among other things, the sociolinguistic issues discussed in Chapter 10. Under this theory, children from lower class homes may have developed a high BISC proficiency but a low CALP, and this low CALP contributes to difficulty in school, where CALP is stressed.

Cummins goes on to define in more detail these two concepts. He proposes a schema which describes language proficiency as measurable along two continua, as shown in Figure 6.3.

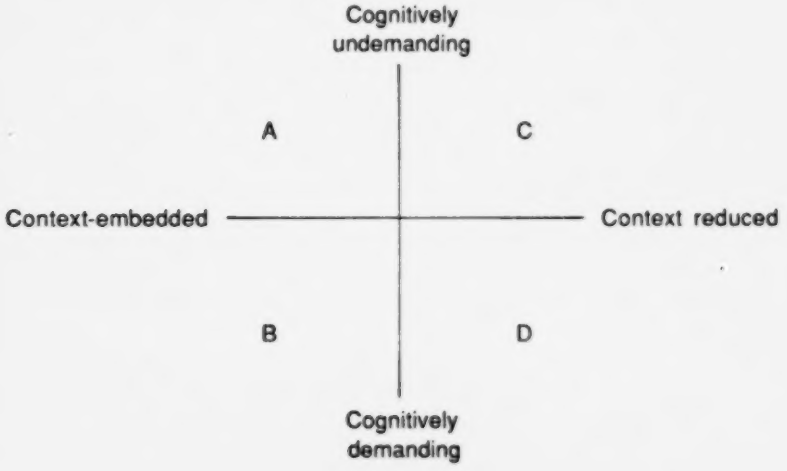

Figure 6.3 Range of contextual support and degree of cognitive involvement in communicative activities.

Thus, children whose language is highly context-embedded may perform poorly in mathematics and other scholastic subjects which demand context-reduced language. A common mistake of teachers is to attribute academic failure to a low position on the vertical (cognitive-demanding) scale, whereas in fact the academic failure is due more to a low position on the horizontal (contextual) scale. A student placed in area B on the above schema may be able to discuss cognitively demanding material only in a context-embedded way, and may be mistakenly placed by the educational system, through its testing procedures, in areas A or C.

Labov (1973) claims that educational tests given in laboratory situations free of context must necessarily measure

skill along the horizontal scale of the above schema, while purporting to measure skill along the vertical scale. This is undoubtedly the best line of attack against Oller's evidence in support of a single linguistic aptitude: an attack on the testing procedure as either testing something other than it purports to measure, or else as not testing all the possible competencies involved in linguistic proficiency.

Labov and the sociolinguists naturally stress the social functions of language, and often point to social factors such as "interests, inclinations, temperament, needs, and motivations" (Fillmore, 1979) which affect the development of language and "presumably had nothing to do with intellectual or cognitive capacity". For example, extroverted children have been observed to acquire spoken language (BICS) more rapidly than introverted ones due to their ability to initiate conversations with other children. As another example (Skutnabb-Kangas & Toukomaa, 1976), an experiment with Finnish immigrant children in Sweden revealed a fluency in everyday spoken Swedish (BICS), but a large deficit in tests at school which measured CALP capabilities.

Because of the difficulty in reliably measuring BICS skills, the evidence for their existence as independent from CALP skills is thin. The factor analytical studies reported above do tend to focus on CALP skills, and generally find a single major factor which accounts for most of the variance in all the subskills. Still, some of them find a second factor which may be related to BICS skills. This is true of the Roberge study reported above. Table 6.4 gives loadings onto the two factors.

Table 6.5 Rotated factor structure matrix

Variable	*Factor I*	*Factor II*
Sex	-.01	.17
Field Independence	.54	-.03
Intelligence	.71	.12
Word Knowledge	.79	.31
Reading	.80	.37
Mathematics Computation	.84	-.24
Mathematics Concepts	.87	-.21
Mathematics Problem Polving	.89	-.20

Roberge called factor I a general intelligence factor, and factor II a verbal ability factor, even though word knowledge and reading

correlated far more highly with factor I than with factor II, thereby illustrating the difficulty of naming factors. It may be that the second factor is something like a BICS factor or a context-dependent factor, although it is difficult to state precisely what it consists of.

Whatever the second factor, it is clear that most of what is tested in school correlates with a major cognitive factor which appears to underlie ability in mathematics as well as language skills.

Finally, it is necessary to reiterate the sociolinguistic viewpoint of Athey (1971) and his school. They believe that reading should not be considered as a 'skill' at all, or even as a bundle of skills:

> In this view, the task of learning to read is not a matter of breaking down the reading task into a number of component skills and determining the order in which these skills should be taught. As in spoken language where the child learns to behave according to a set of induced rules, learning to read means learning strategies for attacking and solving problems of recognition and meaning.

This more holistic view of reading supports the high loadings of reading and other mathematical skills, especially problem solving, onto a single factor, since for Athey, reading is essentially a problem-solving activity. In fact, some researchers have argued that if reading is highly related to problem solving, then perhaps the teaching of mathematical skills, especially problem solving through "inquiry type laboratory and field activities, would seem to be an ideal to promote intellectual growth and thereby language and reading ability."

Correlation of mathematical skills with language skills

The data reported in Table 6.5 are typical of factor analytical studies. Both mathematics scores and verbal scores load heavily onto a single factor. As a second example, Streiff and Oller (1982) gave the California Achievement Test battery to American junior high school students and found a single factor which

correlated at .95 with a cloze test and at .90 with a test of arithmetic computation.

More sophisticated statistical techniques have also been used. Cluster analysis appears to a particularly fruitful method which resembles factor analysis in its partition of abilities into independent clusters, as well as in its difficulty in naming those clusters. A cluster-analytic study by Feagans and Appelbaum (1986), found six categories of linguistic learning disabilities and compared the mathematics achievement of students who fell into the various categories. Significant differences were found. Of interest is the finding that students who could understand and paraphrase narratives performed significantly higher in mathematics achievement than other groups, including a cluster who showed high syntactic and semantic skills. This finding was consistent across American samples in grades 1, 2 and 3.

In general, many, many studies over the past 50 years have found significant correlations between mathematical abilities and language skills. Reading comprehension seems to be the most highly correlated with mathematics achievement. Hinkelman (1956) found correlations of 0.78 (grade 2), 0.71 (grade 5), and 0.77 (grade 7) between reading marks and arithmetic marks. At the secondary level, the correlations decline slightly but are still highly significant. Correlations between .40 and .60 are generally found, for example, studies by Pitts (1952) and by Posamentier (1966). There is even some evidence that the relationship may be partially causal: Call and Wiggin (1966) showed that special classes in reading raised algebra marks, while Finck (1935) found that tutoring in reading influenced achievement in several subjects, including mathematics. On the other hand, Curry (1955) did not find any significant differences between groups with or without special reading instruction, and Russell (1941) found that special practice in reading algebraic materials did not significantly benefit students in solving algebra problems.

To the extent that reading can be divided into subskills, it is possible to measure the correlations between mathematics achievement and scores on tests of those subskills. An early study by Buckingham (1937a) found significant correlations between algebra scores and 'significance of paragraph' (.30), 'predicting the outcome of events' (.41), 'understanding directions' (.38) and 'noting details' (.11). These relatively low correlations, as compared with results cited above, provide evidence that it is a

global reading factor rather than specific skills which accounts for academic success in algebra.

Besides reading ability, other language skills correlate highly with mathematics achievement. Chief among these is vocabulary (e.g. Buckingham, 1937b; Johnson, 1944; Drake, 1940). Listening ability is also a good predictor of mathematics achievement (Murphy, 1963). In general, correlations of under .50 are found between various mathematics scores and various language scores. At least among primary and secondary school children, these results are fairly consistent when measured with first-language testing or second-language testing.

It appears that as students become older, the mathematics-language correlations decrease. Several studies of university students have found quite low correlations. Table 6.6 (Zepp, 1981) shows the results of testing with two years' intakes of university students in Lesotho who were studying mathematics in English, their second language.

Table 6.6 Correlations of various language skills with marks in mathematics

Co-op Tests	1	2	3	4	5	6	7
1. Vocabulary		.54	.68	.44		.36	.28
2. Comprehension level	.57		.78	.12		.10	.08
3. Comprehension speed	.53	.78		.22		.18	.12
4. Expression	.50	.49	.51			.36	.29
5. Total English						.38	.35
6. 1st sem. math.	.04	.01	.32	.12	.06		.83
7. 2nd sem. math.	.00	.01	.02	.03	.00	.83	

CELT	1	2	3	4	5
1. Vocabulary		.53	.39	.27	.20
2. Structure	.53		.62	.39	.30
3. Listening	.59	.53		.29	.10
4. 1st sem. math.	.08	.07	.12		.83
5. 2nd sem. math.	.11	.06	.20	.83	

Two batteries of English tests were given: the Cooperative English Tests (Coop tests) and the Comprehensive English Language Test (CELT). The two years of students are shown above and below the diagonal in the table.

The mathematics-English correlations in the table are all positive but rather low. In fact, results from other second-language studies do not paint a consistent picture of mathematics-English relationships. Jones (1974) reports two studies from African universities which yielded even lower correlations than those shown above. In Nigeria, tests of verbal analogies and of reading correlated with pure and applied mathematics marks as follows:

	Pure Maths	Applied Maths
Verbal analogies	-.08	-.01
Reading	-.01	-.10

Similarly, Jones gave two general English tests to students in Malawi, at Chancellor College and at Soche Academy, and obtain the following correlations with their marks in mathematics:

	Chancellor	Soche
English Test	.07	-.05
Test of English	.18	-.07

Obviously there are many differences from country to country, group to group, and test to test. Moreover, these locally constructed tests have not been checked for reliability. But even from the results of the two years' intake in Lesotho, it appears that the measurement of correlation coefficients can depend on the nature of the group tested. It is therefore not always correct to make general statements about mathematics-English correlations as though they were universally true. One type of student may register a high correlation, while another may yield a low one. It has already been noted that such correlations measured among young children are usually higher than those measured among university students. Murphy (1963) has recorded significant differences between boys (r=.57) and girls (r=.47) in

correlations between English-listening ability and mathematics achievement. Eagle (1948) showed that among students with low mental age, L1 reading speed correlated positively with mathematics success, but among students with higher mental age, the correlation was negative.

A few studies have undertaken to document longitudinal changes in these correlations. At the primary school level, Hinkelman (1956) found relatively constant correlations between mathematics marks and (L1) English marks: .78 in grade 2, .71 in grade 5, and .77 in grade 7. Other data from Lesotho (Zepp, 1982) show changes in correlations over the course of secondary education. Marks in mathematics, science, and English (all done in L2: English) were recorded (Table 6.7).

Table 6.7 Correlations throughout secondary school

Form (secondary school)	1	2	3	4	5
English-mathematics	.56	.37	.54	.29	.18
English-science	.51	.40	.67	.47	.48

Thus, English-mathematics correlations appeared to decline over the five-year period, while English-science correlations remained relatively constant.

The same procedure was carried out with Chinese-speaking secondary school students in Macau (Zepp, 1988), where the English mark was divided into the subcategories of grammar, general English (mainly reading comprehension), composition, and dictation. The results are presented graphically in Figure 6.4, where it can be clearly seen that the correlations decline over the course of six years of secondary education. The strange behaviour of the 'dictation' graph can be attributed to the fact that in form 4 the dictation changes from 'prepared dictation' to 'unprepared dictation'. Dictation changes considerably when the student has had a chance to study the dictated material beforehand, and so the two types of dictation measure very different skills.

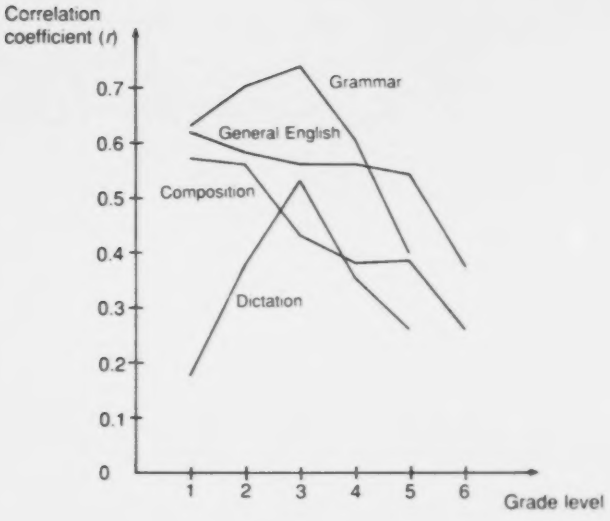

Figure 6.4 Correlations of mathematics and English

We must ask why the English-mathematics correlations decline during secondary school. The hypothesis which may immediately come to mind is that a certain rudimentary level of English is required to learn any subject, including mathematics, since the teacher must be understood, test questions must be interpreted, and textbooks must be read. But by the later years, after English has been studied for some time, most if not all students have mastered these skills to an extent that additional knowledge of English no longer contributes to additional success in mathematics.

In order to test the above hypothesis, we might divide a class into groups of high and low English ability. In the high group, where many basic deficiencies in English have been eliminated, the correlation with mathematics would be expected to be low, according to the hypothesis. Among the students who were weak in English, the lack of basic skills should contribute more heavily to difficulties with mathematics, and a high correlation should be expected.

Such a test was made with the Macau students. In each year of study, the class was divided into two groups of high and low English marks. Figure 6.5 shows the means of the correlations between English marks and mathematics marks for the high English group and the low English group.

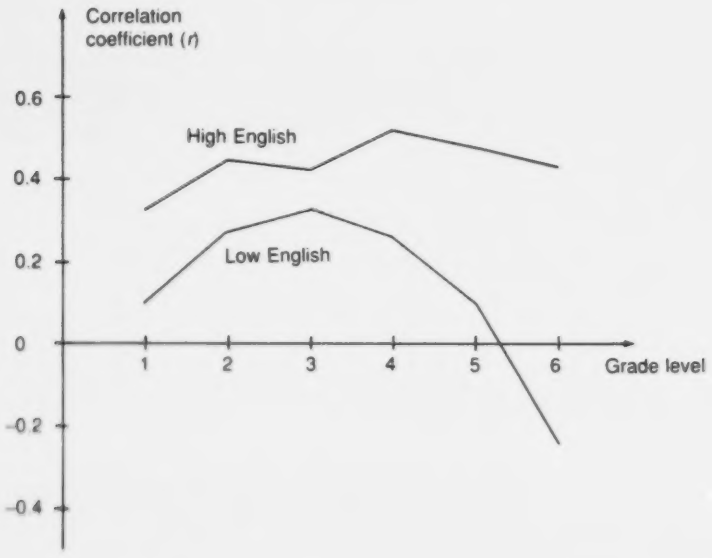

Figure 6.5 Correlations of high and low English groups

The results are quite the contrary of what the hypothesis predicted. It may be that school marks are dependent on many affective factors such as motivation and interest, and these factors may have differential effects for the high and low English groups. Or it may be that the English marks, even those in the same area such as grammar and measured with the same test, measure different abilities among the various age groups. For example, among the low group, students may achieve a few marks from their ability to recognize words and other lower order skills, but for students who have already mastered those basic skills, additional marks in English may be due to skills in verbal reasoning or other skills which correlate more highly with mathematics.

For whatever reason, it is still clear that mathematics-English correlations decline over the course of children's education, from highs of .70 to .80 in primary school falling to lows of .10 to .20 by the university years.

Mathematics achievement of deaf children

Any discussion of relationships between abilities in mathematics and language would not be complete without a treatment of the mathematical abilities of linguistically impaired students. There is a considerable body of research on the abilities of deaf students, prominent among which is the work of Wood and Wood (e.g., Wood *et al.*, 1983). Among 15-16 year-old partially deaf children, they found very low correlations between the *degree* of hearing loss ("and, by implication, linguistic ability") and mathematical attainment. Furthermore, no significant differences in mathematical attainment were found between totally deaf and partially deaf groups. On the other hand, both totally deaf and partially deaf samples scored significantly lower than a normally-hearing control group. The means for the three groups were, respectively, 12.1, 12.8, and 15.5, an indication that deafness is a strong barrier to mathematics learning at those age levels.

Other studies have confirmed the fact that deaf students generally perform worse than the normally-hearing on mathematics tests, but none have led to a satisfactory theory as to why these differences occur. Most puzzling is the fact that even among totally deaf children, there can be found mathematically gifted individuals, indicating that deafness is not a total barrier to mathematics learning. Thus, the Woods argue that "language as 'inner speech' is not a *sine qua non* for computational skill and rational thought." Equally puzzling is the fact that a relatively small hearing loss is a significant predictor of loss of mathematical achievement, but that greater hearing impairment does not seem to heighten the mathematical impairment. There is some evidence to suggest that various obstacles to mathematics learning are simply magnified by hearing impairment, a phenomenon which the Woods term 'multiplication of misery.'

Thus, many questions remain concerning the mathematical abilities of deaf children. The question is by no means a simple one, as the Woods (1984) conclude:

> The nature of the interactions between aetiology, intelligence, sex, family background, learning difficulties, emotional problems and school attended, are extremely complex. To date, no adequate theoretical framework exists to explain the interactions

> between this range of factors. We still cannot explain why deaf children do less well than their hearing peers. (pp. 262-63)

Theories for the correlations

Why, then, should seemingly non-verbal skills like computation correlate so positively with reading ability and other verbal aptitudes? The answer given in the earlier half of this century would probably have been that there is a general entity called intelligence or IQ which explains virtually all variation on scores of tests in schools. Early accounts of intelligence include those of Thorndike (1926), who listed several components of intelligence such as attention, recall and recognition, but who later gravitated towards a more unitary definition of intelligence (Thorndike and Hagen, 1977):

> The ability to see relations in, make generalizations from, and relate and organize ideas represented in symbolic form.

Early statistical research in support of the general intelligence theory was conducted by Spearman (1940), who performed factor analyses on batteries of test scores, and found that most of the variance was attributable to a single factor. He concluded "All branches of intellectual activity have in common one fundamental function." He called this factor the *g* factor, a label which is used in research even today.

Later test batteries attempted to be more comprehensive in their assessment of children's abilities, and not surprisingly found other factors. The Binet test found a general *g* factor plus a specific verbal factor (McNemar, 1942), and Thurstone (1938) identified seven 'primary abilities', which included spatial imagery, perceptual speed, speed of numerical computation, and word fluency. By the 1950s, multiple factor models were in vogue, a trend which culminated in the model of Guilford (1959) shown earlier in this chapter.

Cattell (1963) began the retreat from the many-factor models by proposing a two-factor model of intelligence which consisted of *fluid* intelligence and *crystallized* intelligence. Fluid intelligence is culture-free and possibly innate, and correlates highly with spatial imagery, visualization ability, and creativity.

Crystallized intelligence represents the acquisition of cognitive skills in a specific culture. It correlates highly with both reading ability and mathematical ability. Factor analytical studies which support this two-factor model have been performed by Roe (1951), among others.

In fact, the one-factor model for language proficiency of Oller and the two-factor model of Cummins described above are quite similar to the earlier models of intelligence proposed by Spearman and by Cattell, respectively. In fact, if we accept the possibility that both Oller and Spearman have not used a sufficiently wide-ranging battery of tests, their one factor may be one of the two factors discovered by Cummins and by Cattell. The question is still that of naming or describing the factors. If intelligence tests correlate highly with language tests, should the primary factor be called an intelligence factor or a language factor? The chicken-and-egg situation of whether language ability precedes intellectual ability is one which cannot be well-answered by correlational studies, which are silent on any question of what precedes or causes what.

What is clear from almost all studies, however, is that various abilities in mathematics and abilities in reading all load heavily onto a single factor, suggesting that many verbal and nonverbal skills may be simply surface features of what Oller (1982) calls "a language factor deeper than speech."

The contribution of contemporary sociolinguists is their questioning of labels such as 'intelligence' and 'cognitive ability' for this underlying factor. They suggest that what is really being measured by school tests is the ability to decontextualize a situation. This decontextualization is discussed in Chapters 10 and 11.

Do the answers to the questions raised in this chapter have any effect on what is taught in the classroom? If scholastic attainment is all based on a single factor, then curriculum should be of a general, liberal arts nature, since the development of that general ability could be done in many ways, all of which aim at the training of the mind. On the other hand, if a multiple-factor model is accepted, then education should focus more on individual differences, and individuals should be guided into areas of study suited to their particular abilities, leading to more vocational training.

The common sense approach seems to be that even if there is a common *g* factor, there are other specialized skills necessary for various vocations. It would seem reasonable, therefore, to

insist on a certain training in basic cognition: in 'learning how to learn' before branching out to the other specialized skills. This is not very revolutionary. What must be borne in mind, though, is the notion that mathematics, language, and other nonverbal skills are by no means independent, and that training in the basic 'three R's' may be essentially training in a single way of decontextualized thinking. If so, the conclusion will be that language teachers and mathematics teachers are teaching much the same basic skills.

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CHAPTER 7

THE WHORF HYPOTHESIS

Up until now, we have dealt with studies of mathematics education using English. But we must ask whether the results of these studies apply to other languages, or even to second language English speakers. Is it possible that the relationship between mathematics and language varies from language to language?

Once we begin asking questions of this nature, we must touch upon the deeper question of whether mathematics itself, or even logic itself, varies from language to language. The answer to these questions will have important consequences for the teaching of mathematics to children whose first language is not English.

Benjamin Lee Whorf

Whorf was by profession neither a linguist nor a mathematician nor a teacher; he was a fire prevention engineer who studied linguistics as a hobby. Through his work, he became interested in the effects of language on everyday thinking. For example, he observed that when a drum of petrol was full, people refrained from smoking near it, but when it was empty, people would feel free to smoke, even though it was probably more dangerous to smoke when it was 'empty' because it was actually 'full' of inflammable fumes. The word 'empty' to describe a drum full of fumes, he claimed, carried a connotation of 'harmless'. In another language, one which used a word connoting 'full of fumes' or even 'full of not-petrol', people might be more reluctant to smoke. Thus Whorf became interested in the question, "Do we think in language?", to which his answer was "Yes".

This very simple answer to a seemingly simple question was to have a revolutionary effect on linguistics and even upon philosophy. For, as we discussed in Chapter 1, the prevailing view, which had not changed much since the days of Aristotle, was that thoughts are independent objects which arise prior to language

and are then encoded into language for the purposes of communicating those thoughts. To state that the language actually *is* the thought is a radical departure from the Artistotelian view.

Clearly, speakers of different languages have different world views. We have only to meet traditional Africans, Orientals, Arabs or Indians, in order to be instantly aware of immense differences in thought. Likewise, anyone who has studied a foreign language is aware of the impossibility of exact translation from one language to another. It is often very difficult, even for perfectly fluent bilinguals, to take abstract thoughts expressed in one language and to reformulate them in another language. This fact should be surprising to believers of the old school, since for them, the atom of thought should be expressable or encodable into any language in which the thinker is fluent.

Whorf himself based many of his ideas on his studies of the Hopi Indians in the American Far West. He noted that space and time are not expressed in the Hopi language; rather, the Hopi grammar and vocabulary express ideas using the contrast of particles with field of vibrations. The Western notion of time as a homogeneous and uniformly moving continuum cannot be easily expressed. Instead of space and time, the Hopi use what Whorf referred to as "MANIFEST and MANIFESTING (or, UNMANIFEST) or, again, OBJECTIVE and SUBJECTIVE" (Whorf's capitals). The unmanifest includes not only the future, but also mental activity, feelings and all things connected to the great mystical cosmos.

A related mathematical idea was that Western cardinal numbers sometimes have meanings which are actually ordinal. The phrase 'ten men' has a different meaning from 'ten days', for it is impossible to see ten days at once. Cyclical time such as days, months or seasons, really is an ordinal concept, since the days are in order. In the Hopi language it is impossible to say 'ten days'. The Hopi say 'after the tenth day' and never measure time in units similar to length units. Whorf went on to show just how permeated English is with space-time words to describe almost everything. Consider the following sentence, cited by Whorf (1956) which would never occur in Hopi:

> I "grasp" the "thread" of another's argument, but if its "level" is "over my head" my attention may "wander" and "lose touch" with the "drift" of it, so that when he "comes" to his "point" we differ "widely", our "views" being so "far apart". (p. 146)

How could Western mathematics and physics be invented or even understood by the Hopi? Some mathematical ideas are expressable, but not in terms of the uniform space we tend to think of:

> The Hopi do not need to use terms that refer to space and time as such. Such terms in our language are recast into expressions of extension, operation, and cyclic process provided they refer to the solid objective realm. They are recast into expressions of subjectivity if they refer to the subjective realm - the future, the psychic-mental, the mythical period, and the invisibly distant and conjectural generally. Thus, the Hopi language gets along perfectly well without tenses for its verbs. (p. 64)

Thus, even though cardinal numbers might be expressable in Hopi, the number line and the idea of complete, ordered field, would be difficult to express. The notion of an infinite set, if expressable, would belong to the subjective realm, as would the logical notion of hypothesis or supposition. In which realm would the Hopi place our abstract, axiomatic systems, which we tend to treat as objective objects even though they are highly suppositional?

The point is that mathematics and physics in Hopi must be radically different from ours in the West. This view is summarized by Sampson (1980):

> One might take the Whorf hypothesis to mean that even logic is relative to language, so that, say, if Aristotle had been a Hopi then modern logic as well as modern physics would have developed quite differently. (p.89)

Imagine trying to teach Hopi children mathematics in a Western school! Their ideas would be conditioned by the language they were thinking in, and they would have enormous difficulty adapting to our type of thought.

The following is Whorf's oft-quoted summary of his theory:

> The background linguistic system is itself the shaper of ideas, the program and guide for the individual's mental activity. Formulation of ideas is not an independent process, strictly rational in the old sense, but is part of a particular grammar, and differs, from slightly to greatly, between different grammars. We dissect nature along lines laid down by our native languages. The categories and types that we isolate from the world of phenomena we do not find there because they stare every observer in the face; on the contrary, the world is presented in a kaleidoscopic flux of impressions which has to be organized by our minds - and this means largely by the linguistic systems in our minds. (p. 212)

Actually, part of the credit for this theory must go to Edward Sapir, a linguist and teacher who had an enormous influence on the ideas of his student Whorf, and whose ideas actually foreshadowed those of Whorf. As early as 1939, Sapir (1941) was stating:

> Human beings do not live in the objective world alone, but are very much at the mercy of the particular language which has become the medium of expression for their society. The fact of the matter is that the 'real world' is to a large extent unconsciously built up on the language habits of the group. We see and hear and otherwise experience very largely as we do because the language habits of our community predispose certain choices of interpretation. (p. 93)

These ideas are so close to those expressed by Whorf that the hypothesis is often called the Sapir-Whorf hypothesis.

Thus, mental structures are a function of language. This statement has implications which reach far beyond the teaching of language in Hopi. It implies that even if a Hopi child has learned English as a second language, he/she has already developed a structured way of looking at the world filtered through the Hopi language. To study mathematics in English would be impossible, since the actual cognitive structures are Hopi, not English.

The Whorfian theory is therefore two-fold: not only is thought conditioned by language; once the mental structures are

fixed in one language, they cannot accommodate the structures necessary for thought in another language. This view is very pessimistic, for it implies that no real meaningful communication can exist between people of different mother tongues. It also means that attempting to teach Western mathematics to speakers of non-Western languages is a waste of time: they cannot hope to understand mathematical ideas because their language has shaped their minds in ways that preclude the accommodation of Western ideas. Similarly, their non-Western mathematical systems must forever remain an enigma to us.

Testing the hypothesis

Has the Whorf hypothesis stood up to the test of experimental scrutiny? Early research appeared to support the hypothesis. These studies examined colour coding by speakers of various languages. For example, some languages such as Navajo do not distinguish between blue and green, while others, such as Fulfulde, can describe shades of red much more precisely than English can. But the simple description of colours is not what the Whorf hypothesis is all about; of course, different languages express things differently. Whorf's description of the Eskimo language's many terms for snow simply states that the languages are different and in no way proves that Eskimos perceive snow any differently from speakers of, say, English, who do not have the rich vocabulary for describing snow. The real question concerns the effect of differences in language on behaviour or perception. Brown and Lenneberg (1954) classified colours by degree of 'codability' (short, commonly used words), and showed the many colours to subjects for future recall. It was found that the more codable colours were remembered with greater ease and accuracy. Thus, the language had an effect on the recall, suggesting that memory stores language as the identifying criterion for colours. The experiment was done with English language speakers only, but we must deem it reasonable that in any language, there may be a correlation between the 'codability' of colour and the ability of subjects to remember colours. For example, if the Fulani (the speakers of Fulfulde) have names for different shades of red, while English does not, the claim is that the Fulani will remember the shades as different, while speakers of English will not.

Even this reasonable extension to other languages has been questioned. An experiment by Berlin and Kay (1969) showed that

in several languages, some colours were generally more codable than others in all of the languages. This result suggests that perhaps there are certain basic or 'focal' colours which will be easily encoded, and also easily remembered in any language. If this were true, the correlation between codability and rememberability of colours might not be a direct or a causal one; it might simply result because some colours such as red are universally more codable and easier to remember.

The Whorf hypothesis *is* observable in a very simple way: speakers of African languages which count in base twenty, when asked to count out a large pile of beans, for example, will naturally count out piles of twenty each, thereby facilitating the counting process by twenties. This is clearly behaviour which is dictated by the number system laid down by the language. I have personally attempted to teach illiterate adults of such language groups to read and write numerals in base ten, and it proved a much more daunting task than I would have imagined.

A second way in which Whorfian differences related to mathematics may be readily observed is in the memorization of numbers. It has been shown that English speakers can memorize ordinary base-ten numbers of up to seven digits, on average. However, an artificially produced number system called 'octals', in which each set of three digits is given a separate name, enables subjects to memorize numbers of nearly twenty digits. This result is evidence that the way in which numbers are named affects the ability to remember them, a clearly Whorfian finding.

Another way in which Whorfian differences might be related to mathematics is through the geometric classification of forms. Many languages have 'form classes', according to which all nouns are classified by some form. For example, in English, nouns are countable or non-countable (e.g., 'seeds' are countable, while 'rice' is non-countable, since we may say '5 seeds' but never '5 rices'). In many languages, all nouns must be preceded by a classifier to suggest its shape, as sometimes seen in English (e.g., a stick of dynamite or a loaf of bread). The Whorf hypothesis would predict that languages which have explicit form classifiers should lead to behaviour in which subjects naturally tend to sort objects by shape, especially shapes along the lines of the classifiers. This was actually shown to be the case among Navajo children by Carroll and Casagrande (1958), but they also found it to be equally the case among speakers of English, where classifiers are not so prevalent. Perhaps, like the data on colours, the data on

forms suggest that classification by form is either universal or is attributable to factors other than linguistic ones.

In Chapter 3 we cited a test of concept formation used by Gay and Cole (1967), called the four-card problem, in which subjects were asked to determine which of two pairs of coloured cards typified a rule, which might be a conjunctive one (e.g., red and green) or a disjunctive one (e.g., red or green or both). The authors reported striking differences between American subjects and Kpelle subjects in Liberia, even though the test requires virtually no explicit language. In particular, the Americans did much better than the Kpelle in discovering the conjunctive concepts, while the Kpelle were far superior at discovering the disjunctive ones. These results were given a Whorfian explanation by the authors: the English language expresses conjunction clearly and unambiguously using the word 'and', while disjunction is expressed by the more ambiguous, 'or', which may carry either an exclusive or an inclusive meaning. On the other hand, the Kpelle language expresses disjunction unambiguously, while conjunction is expressed in various convoluted expressions. Thus, Gay and Cole concluded, "The most obvious hypothesis is that the precision of the Kpelle language with respect to disjunction aids them in this task."

Gay and Cole's work is important for this discussion, for it not only supports the Whorf hypothesis; it suggests a simple mechanism by which it works. Their hypothesis is that the more precisely one can express a concept in a given language, the easier it will be for subjects to acquire the concept in that language.

The study just cited is not conclusive, as it provides circumstantial evidence only. Any one of a vast number of cultural differences between Americans and Kpelle could be responsible for differences in results on the four-card problem. The linguistic explanation sounds convincing, but it must be tested further.

In order to test this finding, the experiment was repeated among the Adjukru, who live approximately 350 kilometers from the Kpelle across the Ivory Coast border. Their culture is quite similar to that of the Kpelle, but they belong to a very different language group. The Adjukru language expresses disjunction similarly to Kpelle, while conjunction is expressed much more simply. The results of the experiment are given in Table 7.1. In reading the table, note that a high score indicates a poor performance, since it indicates that the subject continued to make errors after many trials.

Table 7.1 Median trial of Last Error of Kpelle and Adjukru Subjects

	conjunction	*disjunction*
Kpelle schoolchildren	1.5	0.0
Adjukru schoolchildren	2.0	40.0
Kpelle adults	14.0	0.0
Adjukru adults	3.0	6.0

Thus, the Kpelle had no trouble with the disjunctive items, while the Adjukru schoolchildren found them almost impossible, contrary to the expectations of the Gay-Cole hypothesis. The Kpelle adults found conjunction harder than disjunction, as predicted by Gay and Cole.

That the culturally similar but linguistically different groups performed differently is itself support for the Whorf hypothesis, but the data do not seem to fit the Gay-Cole version, since the easily expressable disjunction in Adjukru should have led to rapid acquisition of the disjunctive rules. Moreover, both groups had great difficulty in explaining the rules they had discovered. One might expect them to state immediately, "it's the red one and the green one", for instance, but almost no such explanations were forthcoming.

Other research into the hypothesis is equally circumstantial and speculative. Indeed, the methodological problems of such research are staggering. Any experiment must be carried out in some language, and that interference may affect the results of the experiment. Moreover, no two linguistic groups can be compared *ceteris paribus*, for there will inevitably be differences in culture or environment. Only speculative interpretation of correlational results or other similarities is possible in most instances.

For instance, Hoijer (1953) noted that the Navajo language has only verbs which express involvement or participation of individuals in acts, not initiation of actions. This led to speculation that their language is the reason for Navajo passivity and lack of initiative.

Most of the research during the 1970s failed to confirm the hypothesis. For example, it was noted that many children and adults were able to master complex ideas in a second or third language. It became clear that mental structures acquired in the

first language were not *absolute* obstacles to learning second-language concepts, although it was still debatable whether they provided any hindrance at all. This led to different versions of the Whorf hypothesis. The strong version was the original one, which stated that the first language acted like blinkers to thought in other languages. The weaker versions state that although concepts *could* be learned in a second language, thought still tends to proceed along lines laid down by the first language.

The weak hypothesis

In order to test the weak hypothesis, it is necessary to compare students who are learning in their second language with those who are learning in their first language. All the difficulties of keeping other variables constant remain, and it is therefore extremely difficult to verify or to refute the weak hypothesis experimentally.

Nevertheless, most experiments with second language learners have failed to provide support for the weak hypothesis. We have already described the 'hidden card' experiment in logic, wherein subjects are asked to draw inferences, if possible, about the hidden side of a card based on statements using logical connectives.

Zepp (1982) compared performances of Sesotho-speaking secondary school students on the test given in Sesotho with results of the test given in English. Not only did the Sesotho group and the English group perform roughly the same at lower grade levels, the *English* group performed significantly better in form 5. Thus, students performed better in their *second* language than in their first by the time they had reached form 5. This result contradicts expectations of the Whorf hypothesis, which would predict that the group doing logical exercises in their second language (English) would not be able to master the logical difficulties as well as they could in their first language (Sesotho).

The experiment was repeated (Zepp, 1987a) with Chinese-speaking university students, the test being administered in English to one group, in Chinese to another, as well as in English to a comparable group of New Zealand students whose first language was English. No significant differences in groups were found. Moreover, all groups made roughly the same *types* of errors. Thus, there was no evidence that logic done in the second language (English) was any different from that done in the first language

(Chinese), nor that first language English speakers' logic was any different from that of the group whose second language was English.

Even if subjects in the above experiment showed no differences in their modes of thinking, the question remained whether students can learn logical principles in their second language. The experiment was repeated, (Zepp, 1986) but this time, after giving the initial test in English to the Chinese students, the subjects were given 20 minutes' instruction on some of the logical principles, such as how to reason from the contrapositive, or the fact that reasoning from the converse or the inverse in not logically valid. One week later, the test was given, again in English, and the results were strikingly better. There was no question that the students had learned the logical principles in their second language. Thus, there was no evidence of any Whorfian barrier to learning logic in a second language.

A third question arose as to whether students who had learned logical principles could then apply those principles in Chinese-language situations. In order to examine this question, the experiment was repeated (Zepp, 1987b), but this time, both the pre-test and the post-test were administered in Chinese, while the instruction session was again given in English. Again, the instruction period produced very large increases in scores, equal to the gains achieved when the pre-test and post-test had been given in English. Thus, the knowledge gained in the second language was apparently completely transferable to first language situations. In all, none of the experiments in the series produced any evidence in support of the Whorf hypothesis, even the weaker version.

By the 1980s, the Whorf hypothesis had been so weakened by the lack of experimental support that most linguists and educators no longer took it very seriously. Statements were being made along the following lines: (Rosch, 1977)

> At present, the Whorfian hypothesis not only does not appear to be empirically true in any major respect, but it no longer even seems profoundly and ineffably true. (p. 519)

Even more damningly, Macnamara (1970) argues:

> We must visualize the human language user as a far more dynamic agent in his approach to speech than either Whorf or bilingual theory builders seem to imagine. Moreover, it seems likely that linguistic processes are only a small part of the cognitive functioning which is associated with either the production or the interpretation of speech. I do not believe that there is any evidence to justify claims of Whorfian relativism. (p. 36)

Attempts at revival

By the 1980s, it looked as though both weak and strong versions of the Whorf hypothesis had been relegated to the status of historical curiosities. But then a few studies started turning up evidence that language and thought were interacting in ways which had not yet been considered.

One of the chief findings of this nature was that of Dawe (1983), who measured the correlations between ability in mathematical logic and linguistic competence in both first language and second language. His logic test was given in English, the second language of the subjects. Correlations of the logic test were higher with first language competence than with second language competence, even though the logic test was in the second language.

This finding is evidence that first-language structures influence thought processes even in the second language. It also raises the question of whether students studying in their second language perform their thinking in their first language. If so, it would support the *weak* Whorf hypothesis, since the students were *able* to learn the logic in their second language, but were inclined to think in their first language. It would be an unpleasant finding, however, if it were ascertained that students around the world who were studying mathematics in a second language were in reality merely translating the lessons into their first language in order to perform the mental operations.

Apart from correlational studies such as the one just described, an increasing number of educators from third world countries are again raising questions concerning the adaptability of mathematics to speakers of non-Western languages. We shall examine this topic in much more detail in Chapter 11,

Ethnomathematics. For now, it suffices to point out that speakers of, say, African languages, may have first languages which express mathematical ideas very differently from European languages. African students may have difficulty with mathematics because they find the language of mathematics different from their own.

Kazadi wa Mashinda (1988) has explored some of these difficulties in Zairean languages, in pointing out that "We often tend to mistake pure mathematical incomprehensions for those that are linked to language." As an example, he cites a teacher who was attempting to teach the concept of 'greater than or equal to' in French to a speaker of Lingala. The teacher drew a time line with the words "yesterday, today, and tomorrow" in order to illustrate the order relationship. But in Lingala, as in many languages, the words for yesterday and for tomorrow are identical: 'lobi', and the meaning can only be found in the context of the sentence, mainly from the tense of the verb.

This example resembles the many linguistic problems reported fourteen years earlier at a pioneering *Symposium on Linguistics and Mathematical Education*, held in Nairobi in 1974. All such studies worked on an assumption similar to the Gay-Cole hypothesis that the ease of expression of a concept determines the difficulty of its acquisition. The assumption was that if students from one language group were experiencing difficulty with a mathematical concept in a second language, the structure of their mother tongue could be examined to find reasons for the difficulty. For example, Morris (1974) argued that the Igbo of Nigeria may have difficulty with improper fractions studied in English, because their mother tongue has no convenient way of expressing improper fractions.

The many papers of the nature just described never subject their claims to the scrutiny of rigorous testing. They must therefore be treated as merely speculative. In fact, as we have seen, the experimental data from many studies suggest that students *can* learn mathematics in a second language and that the logical difficulties are roughly the same across languages.

Of course, if students cannot speak English at all, they will have enormous difficulty learning mathematics in English. This is no indication of Whorfian differences. Thus, the studies of Morris and of Kazadi may not be about Whorfian differences at all; they are simply saying that students with a very low competence in their second language will have difficulty learning mathematics in that second language, a statement which no one can deny.

The reality of the situation in developing countries is, however, that students often must study in a second language in which they are much less competent than the teacher expects. Teachers going to developing countries for the first time must face this fact and must adapt their language accordingly. Above all, they must be sensitive to subtle differences in language such as those described by Kazadi. Kazadi is correct to criticize teachers for not being aware of major differences between languages. What is even worse is for local teachers who speak the first language of the students to make unwarranted assumptions about the students' abilities in their second language. Too often, highly educated locals return to the classroom with an attitude of, "If I could master difficult mathematics in English, then my students should be able to do it too," and they often use a language incomprehensible to their less linguistically competent students.

But these teachers must not take the pessimistic stance that the students' linguistic difficulties are Whorfian: because of their first language, they will not master the mathematics no matter how good their second language is. For students with poor second language skills, mathematics teachers must be language teachers as well, for they must boost students over the linguistic hurdles before they can study mathematics in their second language.

The present situation

At present, almost no one gives much credence to the strong version of the Whorf hypothesis. The weak version is not one hypothesis, but a continuum of views on the degree of influence which language has on thought. Most experts today express belief in some degree of the weak version, but this belief is usually vague and has few educational consequences. We have seen several rather common sense examples of how language clearly affects thought, for instance in the effect of number-base systems on counting.

Another effect which has been documented (Glucksberg and Danks, 1985) is that the names of objects can influence the way in which they are used. For example, the English word 'screwdriver' suggests that the object is used to turn screws. In experiments where a screwdriver may be put to other uses, subjects may find it difficult to discover the alternate uses because the name 'screwdriver' fixes the use to which it must be put. In another language, the word for the tool might be equivalent to 'metal rod', and that term may tip the user off as to alternate uses.

As demonstrated by the experiments on colour, language may aid in the memory of colour, although there is evidence that some colours may be naturally easier to remember than others. Similar experiments with shapes have shown that language can have an effect on how shapes are remembered. For example, given the following shape,

subjects in one group were told that the shape was a dumbbell, while subjects in another group were told that it was a pair of spectacles. In later recall tests, the dumbbell group and the spectacles group tended to reproduce the shape differently:

dumbbell

spectacles

It is clear that language facilitates memory, but that it can also distort memory. The brain may sacrifice a bit of accuracy in order to gain ease of recall. If one language describes a shape using a word which suggests some object, such as 'dumbbell', and if another language uses a different word, equivalent to 'spectacles' then Whorfian differences may occur in the subsequent recall of the shape.

A similar instance of the effect of language on memory was reported by Loftus (1975). Eyewitnesses to an automobile accident were asked at what speed the car was travelling when the two vehicles crashed. To other groups the question was phrased identically, except that the word 'crashed' was replaced by other terms, such as 'bumped', which have a gentler connotation than 'crashed'. The recollections of the speed of the car were significantly higher when the word 'crashed' was used. The word appeared to suggest a higher speed than the word 'bumped'.

Perhaps the area in which language may have the greatest effect is in its grammar rather than its vocabulary. For some grammars stress certain elements, thereby calling attention to them. For example, many language, such as Japanese, have an array of pronouns and forms of address by which the relative social standing of all parties involved is always made clear and

explicit. It might be argued that the Japanese maintain a high degree of awareness of social ranks *because* of their language. Similarly, African languages which describe kinship relations precisely, such as 'third older brother' or 'brother who follows a sister' may lead to a heightened awareness of kinship relations. Perhaps those mathematical puzzles about the son of an aunt or a grandmother are trivially easy to Africans using their own languages.

But the cause-and-effect relationships of these examples are extremely difficult to document. Perhaps the Japanese awareness of social standing caused and now perpetuates a language which reflects that awareness, rather than vice versa. Perhaps Africans, in their awareness of kinship relationships, invented languages which reflect those relationships. It is very much a chicken-and-egg situation, where both language and social custom tend to reinforce each other. These are examples of the notion that language, culture, and thought are very much interrelated, and that it makes little sense to ask which comes first, the language or the thought. The fact is, they perpetuate and complement each other. Whorf (*op. cit.*) himself was aware of such difficulties:

> Which was first: the language patterns or the cultural norms? In the main they have grown up together, constantly influencing each other. But in this partnership the nature of the language is the factor that limits the free plasticity and rigidifies channels of development in the more autocratic way. This is so because a language is a system, not just an assemblage of norms. (p. 156)

Much of Western mathematics, especially calculus, is based on a time-line concept which reflects our notion of time. From any 'point' in our linear, one-dimensional time, we may look towards the future or to the past. This reflects our verb tense system: present, past, and future. Many languages of the world, if not most, do not have a verb structure similar to the English tense structure. Many focus on 'aspect' rather than tense. The aspect of a verb tells whether an action has been completed or repeated, punctual or lasting. For example, speakers of such languages, when about to leave, may say something which sounds to us like "I have left", whereas we would naturally say "I am about to leave." This is because the act of leaving is a once-only, completed action and requires the completed aspect rather than a future tense.

Speakers of aspect languages can still learn Western calculus, and they can learn to speak English. Still, a weak Whorf hypothesis might predict that in problem-solving situations these people may tend to focus on aspect rather than on tense, or otherwise might deal with time in a different way. We must still ask whether a view of time is instilled more by the surrounding culture than by the language, and once again, the answer is that language and culture cannot be separated, and that the two together may cause a speaker to pay attention more to one facet of a situation than to another.

As in the question of colour, there is some evidence that the notion of time precedes language. Weist (1985) argues from experiences with very young children that:

> Children can think abstractly about events in time a full year (or more) earlier than the most precocious of prior estimates. The concept emerges well before the language. (p. 206)

But in any case, imagine the richness of thought of a bilingual person who can view time in both or several ways. As we shall discuss in the Chapter 8, there is a sort of synergy at work, where the two views of time can produce a higher order mental structure in which the two time-views may be compared and analyzed.

Whorf's legacy

While the Whorf hypothesis has been largely discredited in its stronger forms, linguistics owes a tremendous debt to Benjamin Whorf. For it was he who opened up the train of thought which allowed linguists to escape from the old Aristotelian dichotomy between thought and language. By stating that language precedes thought, Whorf challenged the conventional wisdom of the day, that thought is independent from and precedes language. Since that questioning of tradition, the study of language has never been the same.

Linguistics today has not resolved the question entirely. There is a realization that thought and language cannot be separated, but that they are not identical, either.

Whorf's writings make very worthwhile reading, for they are clear and to the point. It is impossible to read Whorf without

developing a deeper appreciation for the cultural and linguistic differences between English and Hopi. The Hopi really seem to think in a very different way from Westerners , and difference between languages is one plausible explanation for those differences in thought.

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CHAPTER 8

BILINGUAL EDUCATION

In this chapter we discuss the effects of bilingual education on mathematics learning. In particular, are bilingual children placed at any advantage or disadvantage to monolingual children in the mathematics classroom?

We must first address the question, "What is bilingualism?", which subsumes another question: "What does it mean to be competent in a language?" There are no easy answers to these questions. Speaking a language is not a zero-one process that someone either can or cannot do, but rather a continuum of abilities. It is always possible to improve one's language skills.

Even if we try to define fluency in terms of ease of communication, or the ability to express nuances, we run into trouble describing people who are 'fluent' in some situations but not in others. Many mathematicians are experienced at delivering very polished lectures in a second language, but are totally at a loss in everyday situations such as talking about repairing their car. The complexity of the problem is made clear by Gaardner (1970):

> There is a common belief that the person who speaks two languages can say anything in one that he can say in the other. That is simply not true. The most common situation finds such a speaker facile in one or more 'domains' of usage in one language, and in other domains in the other. Arithmetic in one. Professional matters in only one. Everyday affairs can probably be conducted in either, but no one can stand up and invent authentic translations of mathematics teaching terminology. (p. 167)

Types of bilingualism

One educational classification scheme (Fishman, 1976) focuses on the educational objectives of bilingualism in schools. The four categories are:

1. *Transitional bilingualism.* In many African countries, for example, the ultimate goal of schooling is mastery of the former colonial language, such as English or French. But children who cannot speak any French must attend the first few years of school in their mother tongue, which may not even be a written one. The study of the mother tongue is therefore just a transitional step towards bilingualism: an uneven bilingualism in which the true focus is on just one language.

2. *Monoliterate bilingualism.* In many countries, local languages are not considered true languages for writing. Even though classes are taught in one mother language or in two languages, all reading and writing are done in the second language.

3. *Partial bilingualism.* Schools may promote speaking and reading in both languages, but there is a separation into subject matter. Mathematics, science, history may be taught in the second language, but there is a strong cultural component of, say, vernacular literature taught in the local language, as is the case in many Middle East countries. This type of bilingual education is used where it is desired that children receive a Western education, but that they should be made aware of their cultural and linguistic heritage.

4. *Full bilingualism.* This term refers of course to a commitment that students should be able to read and write both languages in a full range of situations and contexts, as in Canada.

Since we concentrate on the cognitive development of bilinguals, especially their mathematical skills, it is necessary to mention cognitive classifications of bilingualism. This has traditionally been done using the notions of coordinate, compound, and subordinate bilinguals, as described by Weinreich (1953). A *coordinate* bilingual is a person who has acquired a concept with an attached word in one language, and who has also learned a similar but not identical concept with its concomitant word in another language. A *compound* bilingual has learned one concept, but attaches a different word to it in each language. A *subordinate* bilingual has learned a concept and its word in one language, and has then learned a translation for that word in another language. In some sense, then, the coordinate bilinguals are the true bilinguals, because not only the words, but also the underlying

concepts tend to differ across languages. Weinreich devised the following schema to describe the three categories, using the Russian equivalent of the English word 'book', *kniga*.

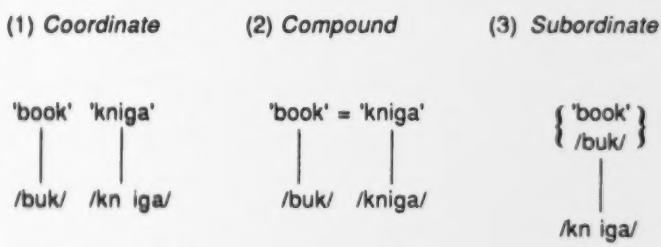

Figure 8.1 Classification of types of bilingualism

These same distinctions can be applied to non-linguistic cases. In science, for example, I may have learned the Fahrenheit scale of temperature in America, but the Celsius scale in Asia. When in America, if someone says "it is 86 degrees", I know exactly how hot it is, but if he says, "it is 30 degrees Celsius", I am no longer sure; I can only relate that figure to my experience in Asia. On the other hand, I know exactly what 30 degrees means in Hong Kong, but would be unsure of how hot it was if the weather report said the temperature there was 86 Fahrenheit. This is an example of coordinate bilingualism in science.

Similarly, a child may be familiar with the metric system in a science class, and may be able to estimate the length of the physics lab in meters, but in a baseball stadium he would estimate the distance from home plate to the fence in feet, and might have no idea how many meters that was.

Most children are subordinate bilinguals when it comes to units of measurement. They 'think' in one unit, but must use a conversion factor to translate to another system. They have no intuitive feeling for metric units, for example, but are taught the usual multiplications for converting metres to feet. This does not constitute a real knowledge of the metric system, just as learning that the French word for 'to love' is 'aimer' does not constitute a real knowledge of French, since the correspondence between the usages of 'to love' and 'aimer' is not one-to-one.

Mathematics teachers should think about this distinction when teaching correspondences between systems, such as units of measurement. Are they teaching a subordinate system, and if so, is

it really aiding the mathematical development of the child to do so? It would be better to aim for a coordinate or even a compound understanding, in which the children can feel comfortable in either system, but can move freely from one system to another.

Little has been said in recent years about the coordinate-compound distinction. This is probably because no pairs of concepts ever exist in a totally coordinate way, nor in a totally compound way. For even if an individual has learned a concept like 'to love' in one language and its near equivalent 'aimer' in another, he must surely realize that there is some overlap or similarity between the two. The two concepts are not mutually exclusive, so the bilingualism is not entirely coordinate. On the other hand, if the individual has mastered both languages, he will realize that the correspondence is not one-to-one, and therefore one set of circumstances cannot be serviced equally by two words: the bilingualism is not compound. The evidence seems to suggest, therefore, that there is considerable correspondence between concepts learned independently in two languages, but that the correspondence is not an identity.

Another distinction is often made between *additive* bilingualism and *subtractive* bilingualism. In the former, the second language or L2 is learned at no expense to the competence in the first language or L1. This is often the case when the child continues to speak and to learn the L1 in many circumstances. On the other hand, students from minority language groups which do not enjoy a high prestige may tend to lose their L1 ability as the L1 is replaced in daily life by the L2. This is a case of subtractive bilingualism.

The additive-subtractive distinction is important to educational theory because it may be related to additive or subtractive cognitive development in the child. It may be the case, as we shall see, that additive bilinguals also experience an additive effect on intelligence, while subtractive bilinguals experience a negative effect. Thus, children may be better able to learn mathamatics in either L1 or L2 if they are fully competent in both languages. This question has been the subject of much research in the past decade.

A final distinction has already been discussed in Chapter 7. That is the distinction between BICS (basic interpersonal communicative skills) and CALP (cognitive/academic language proficiency). It is possible for students to have acquired BICS proficiency in a language without mastering CALP proficiency, or vice versa. When we discuss the effect of bilingualism on cognitive

development, we must be sure of whether we are talking about CALP bilingualism or BISC bilingualism or both.

Bilingualism and cognitive development

During the first half of this century the conventional wisdom about bilingual children was that the additional burden of having to process information in two or more languages was 'confusing' and thus mathematical learning, for example, would not proceed as rapidly as in the monolingual case. Indeed, research showed that bilingual students from minorities such as Pakistanis and West Indians in the U.K., or Hispanics in the U.S. usually fared poorly in comparison to monolinguals there. A review of the early literature by Arsenian (1937) found that 60% of the studies had found major negative effects of bilingualism, with another 30% finding minor disadvantages. There developed a theory of linguistic deficit, whereby these bilinguals were considered to be at a cognitive disadvantage to monolinguals. If they performed poorly in mathematics, it was because their lack of language skills led to a lack of intelligence.

Curiously enough, intelligence was often measured by English language testing, so it was not surprising that children who were weak in English scored poorly on intelligence tests. The testing was done in 'standard' (= academic) English, and most of the bilingual children spoke a very different variety of English from that of the tests. Moreover, the bilingual children tested were usually from working-class backgrounds, since most immigrants were of the lower end of the social spectrum. Since recent data have shown that socioeconomic status is a strong predictor of academic success and even of results on intelligence tests, it is not surprising that immigrant children, usually of low socioeconomic status, performed poorly on intelligence tests. Even Margaret Mead (1927) concluded that bilingual children in Italy were less intelligent than monolinguals.

This may be a case where working class or immigrant children had acquired BISC proficiency without ever mastering CALP skills. Intelligence tests may be simply measuring CALP skills and therefore a result such as the Mead finding may simply indicate only that Italian BISC bilinguals were not CALP bilinguals.

The conventional wisdom of the early twentieth century is still strong today in the minds of many working-class families. A survey of Welsh families by Harrison, *et al.* (1981) showed that

it was precisely the working-class families who did not wish to raise their children bilingually, for fear that bilingual children would be inferior in English.

While the belief that bilinguals are intellectually inferior persists in some quarters, the general view among academics has changed, especially since the milestone work of Lambert and Peal (1962). This was the first study to find positive consequences of bilingualism. One of the reasons for these differences in findings from the earlier studies was that factors such as socioeconomic status, age, and degree of bilingualism were controlled. They found that bilinguals outperformed monolinguals particularly in 'cognitive flexibility', which we may take to mean something akin to creativity, divergent thinking or lateral thinking. They also reported differences in concept formation, manipulation of figures, and completion of figures.

During the 1960s, research hesitatingly followed the lead of Lambert and Peal. Even neutral results, in which no significant differences were recorded between bilinguals and monolinguals, were considered as a good sign for bilingual education in light of its hitherto bad reputation. A study by Lambert and Macnamara (1969) examined first grade Canadian bilinguals' achievement in mathematics, and concluded happily that the experimental group performed as well as the control group in mathematics. On the other hand, the experimental bilinguals showed lower IQ scores (as measured by Raven's Progressive Matrices) than one of the English-only control groups:

> [The bilingual children] are able to follow directions and work out practice problems presented in either language as well as both types of control. There is some evidence for a relative suppression of intellectual development in the experimental class, but not one that can be unambiguously attributed to their type of academic training. (p. 95)

A larger study of Canadian children, known as the St. Lambert Experiment, received substantial publicity (Lambert and Tucker, 1972). The experimental group scored well above average - beyond the eightieth percentile - on national norms of mathematical achievement, as well as the monolingual groups. This time there were no adverse effects on intelligence scores:

> One can be confident, then, that the children have been as able as the French control groups to grasp, assimilate, and utilize mathematical principles taught through French exclusively, and that they have transfered this knowledge to English. By the end of grade IV, [there were] no signs of any intellectual deficit or retardation. (p. 205)

Since 1962, many other studies have verified Lambert and Peal's results, especially in the field of cognitive flexibility. As examples, we may cite a study of Swiss children by Balkan (1970), a study of Afrikaans-English bilinguals in South Africa by Ianco-Worrall (1972), and a study of Hebrew-English bilinguals in Israel by Ben-Zeev (1972). Typical tests of divergent thinking ask subjects to name as many uses as possible for some object shown by the experimenter. A large number of uses cited by the subject is taken as a high measure of divergent thinking. Tests of this nature have been administered to bilinguals by Cummins and Gulutsan (1974) in Canada, and by Torrence (1970) in Singapore, with findings of higher divergent thinking in bilinguals.

It is relatively easy to explain the advantages of bilingualism for flexibility of thought. The theory is that children who learn two languages have two world-views from which to consider any problem. To a given stimulus they may respond in a manner typical of either language, or they may even use similarities or differences between the two languages to arrive at responses which are typical of neither.

Indeed, we have seen in our discussion of concept formation that the more multi-tiered the hierarchy of concepts, the greater is likely to be the ability to view concepts from many angles. Bilinguals naturally have an additional level of concept attainment, that is, the similarity between concepts in two or more languages. This additional level of hierarchical structure may be particularly important for mathematics learning.

Monolingual children who learn the word for, say, an apple, will perceive a one-to-one correspondence between the word 'apple' and the object. They are likely to confuse the word with the reality, that is, to confuse a 1-1 correspondence with outright equality, and believe that there is an essential 'appleness' about such objects. On the other hand, bilinguals will learn two words for the same object. They will realize that the word is not a property of the object, but is rather an arbitrary designation

invented by people to refer to that object. Whether the object is an 'apple' or a 'pomme' is totally dependent upon the linguistic system in which one is working. This separation between the word and its referent is essential for mathematics learning.

For students of mathematics must understand that definitions and axioms are arbitrary. It is an important and essential step in mathematical understanding to realize that Euclid's parallel postulate is not a statement of fact, but depends upon the axiomatic system in which one is working. Such an advantage to bilinguals has been recorded by Ben-Zeev (1972):

> Bilinguals become aware of their languages as internally consistent systems more than do other children. The bilingual might grasp general rules and extend them more quickly.

A follow-up study (Ben-Zeev, 1977), focused on children of various social classes, and concluded:

> The effort to become bilingual results in greater ability to process systemic structures, even for bilinguals in relatively disadvantaged circumstances.

Not only did the bilinguals process structure better, they showed a willingness to reorganize their thoughts when a direct translation was not possible. This ability to reorganize may be an important factor in problem-solving activity, as discussed in Chapter 5.

Another advantage to bilingualism may be that bilinguals will naturally compare the languages they speak, and in order to do so, they must learn to think abstractly about language. Monolinguals are often unaware of the grammatical structure of their language, but bilinguals may tend to notice differences such as word order or tense structure and thereby can analyze one language from without rather than from within. The 1977 study by Ben-Zeev found precisely that: Hebrew-English bilinguals were better than monolinguals at the analysis of sentence structure. It is this ability to distance oneself from a concept in order to analyze it abstractly, which is important for mathematical analysis. Bilinguals may therefore be better able to engage in this abstract analysis. It is precisely this ability which mathematics education should stress when teaching number systems in bases other than ten. By appreciating differences between number-base systems, it is expected that children should

be able to analyze base-ten numbers in a more abstract light, and to understand better the structure of our own number system.

Research has tended to support these ideas. For instance, a study by Liedke and Nelson (1968) found that the concept of measurement was at a more advanced state in bilinguals than in monolinguals. This result could be expected because the easy flow between two language systems might facilitate the flow between two systems of measurement. Bilingual children would readily grasp the notion that an object's length in inches is not really an intrinsic property of the object, but that the choice of unit is relative to the measurement system.

Similarly, a study in the U.S.A. by Feldman and Shen (1976) investigated whether children would still recognize familiar objects which had been transformed by folding or denting. They found that bilinguals were superior at such recognitions. Just as an object is still the same object to bilinguals whether it is called 'an apple' or 'une pomme', so the geometric transformation does not change the intrinsic nature of an object. Along the same lines, Piagetian conservation studies by De Avila and Duncan (1979) found that bilingual first and third graders conserved more often than monolinguals.

Thus, the more recent research has provided evidence that bilingual education *can* have positive effects. Of course, not all experiments have found significant differences. Many have found none at all, and a few have even found deficiencies among bilinguals. A study of Irish-English bilinguals by Macnamara (1966) showed that the bilingual children lagged far behind their monolingual counterparts in arithmetic problem solving, although no significant differences were found in mechanical arithmetic. Negative findings have also been reported from Japan (Tsushima and Hogan, 1975) on verbal skills and on overall academic achievement, as well as from Singapore (Torrence *et al.*, 1970) on creative thinking. These studies can, however, be criticized on methodological grounds (see, e.g., Cummins and Swain, 1986).

All the studies cited so far have used the term 'bilingual', but as we have seen, the term carries many meanings. Which kind of bilinguals are being studied? Most of the positive results have been recorded among additive bilinguals, that is, children who continue throughout their L2 education to maintain or improve their L1. The negative results have usually been obtained from subtractive bilinguals.

In order to investigate possible differences between additive and subtractive bilingualism, Kessler and Quinn (1982)

performed experiments in American science classes using both types of bilinguals as subjects. One of the criteria for evaluation was the ability to formulate scientific hypotheses. Since previous research (Quinn and Kessler, 1976) had shown that syntactic complexity is a strong predictor of the ability to form hypotheses, this variable was also measured in the 1982 experiments. The subtractive Italian-American bilinguals performed significantly higher ($p<.01$) on both hypothesis quality and on syntactic complexity, although no differences in overall IQ were found. For the Spanish-English additive bilinguals, the experiment was slightly different: experimental groups were given instruction in hypothesis formulation and were compared with control groups who did not receive such instruction. Results were much more striking, especially from the experimental groups: the bilingual group more than tripled the mean score of the monolingual group in hypothesis formation, and were far superior in syntactic complexity. Among the controls, the bilinguals were slightly superior in hypothesis quality, but significantly inferior in syntactic structure.

The Quinn and Kessler result has important educational implications. For even if additive bilinguals are slightly weaker than monolinguals when given no instruction, they benefit from instruction to a far greater degree. Thus, additive bilinguals have an advantage in science classes when it comes to the formulation of scientific hypotheses. This finding supports the theory described above that if children are fluent in two languages, they will appreciate the relative nature of linguistic labels and may more readily grasp the relativity of various axiomatic or hypothetical systems. They will understand a hypothesis as a conditional construct rather than as a statement of absolute fact.

Quinn and Kessler described their findings in slightly more Piagetian terms. Instead of stating that monolinguals viewed the 'what if' of science problems as mere statements of fact, the authors concentrated on the fact that the bilinguals' sense of relativity created a disequilibrium which, according to Piaget's theories, was conducive to the accommodation of new knowledge. Piaget claimed that in order to assimilate knowledge, the mind must raise a question or otherwise *want* to acquire the knowledge. The disequilibrium caused by the relativity of the two languages therefore may enhance the assimilation of knowledge in the classroom. Thus, what was termed 'confusion' 30 years ago is now given the positive term 'disequilibrium'.

The threshold hypothesis

If it is true that subtractive bilingualism has neutral or even adverse effects in the classroom, and that additive bilingualism has positive effects, there must be some level of bilingual development at which the negative effects become positive. Such a threshold level between positive and negative effects leads to the threshold hypothesis: that positive cognitive effects result only when a child has achieved a certain level of competence in both L1 and L2 . This hypothesis has been explored by Cummins (1978), who believes the threshold level of competence is "an ability to use language effectively as an instrument of thought and to represent the thinking processes by means of language". The following is the original statement of the hypothesis by Cummins and Gulutsan (1974):

> the threshold hypothesis assumes that when a certain level of competence in two languages has been attained there are aspects either of a bilingual's present access to two languages or of his bilingual learning experiences which can positively influence his cognitive functioning.

Of course in going from positive effects to negative effects, there may be two threshold levels: one necessary to move from negative effects to no effects, and another to move from neutral to positive. Below the first threshold, the interference between the two languages may indeed cause the outright 'confusion' reported in the earlier studies. Available data do not appear to contradict the double threshold hypothesis (Cummins, 1979). Figure 8.2 shows how the double threshold may operate.

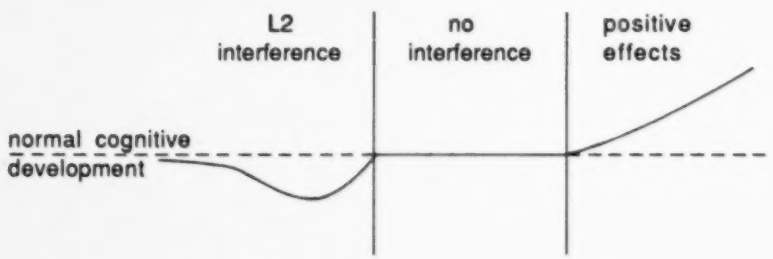

Figure 8.2 The Double Threshold Hypothesis

A related hypothesis proposed by Cummins is often referred to as the *developmental interdependence hypothesis*, which states that a child's development of skills in his L2 will be affected by skills already developed in his L1. "Where the mother tongue is inadequately developed, the learning of an L2 can hinder the continued development of the first, and the inadequate development of the L1 will in turn impede the development of the L2."

The finding of Dawe (1973) that mathematical logic, even when done in the L2 (English), correlates more highly with L1 ability than with L2 ability, lends strong support to this hypothesis. For it would appear that underdevelopment of the L1 hinders the development not only of the L2, but even more so the development of skills learned in the L2.

Other partial evidence is provided by a study by Cummins *et al.* (1984) of Japanese-English bilinguals in Canada. The subjects were given a battery of tests in language competence and other cognitive skills, both in Japanese and in English. The Japanese results were first analyzed by a factor analysis in order to group skills into broader measures of competence. Then correlations were calculated between various scores from the English medium tests and the three factors obtained from the Japanese tests. Significant correlations were reported between the Japanese factor called 'academic proficiency' and similar cognitive/academic measures in English. Thus, cognitive/academic development in Japanese was related to cognitive/academic development in English.

All this discussion of additive and subtractive bilinguals, and of adequate or inadequate development of L1 or L2, is based on *products* of education: an additive bilingual is the *result* of education. Little has said about how these results come about. What educational factors cause children to become additive or subtractive bilinguals, or to develop adequate L1 or L2 skills? Let us turn to the types of programmes which purport to address the needs of bilingual students.

Immersion versus submersion

These two terms are widely used to distinguish between two types of bilingual education programmes. Traditional programmes for minority students who speak a language other than that spoken in most schools (e.g., Spanish speakers in an English medium

school), attempted to single out the underachievers and to give them special intensive instruction in the target language. Such programmes are now called *submersion programmes*. They tend to emphasize the linguistic deficit of the students in the target language, and to de-emphasize the children's L1. Thus, the actual *objective* of these programmes is a subtractive bilingualism: the sooner the children foresake their minority language for the mainstream language of school and society, the more likely they will be to succeed in school, employment, and society.

Immersion programmes, on the other hand, aim for additive bilingualism. The typical programme takes speakers whose L1 is already the dominant language, and immerses them in a second language whose acquisition is socially or politically desirable. The most well-known of these programmes are found in Canada, where English speakers are immersed in French language instruction with the objective of rendering them completely bilingual. Not only do students study both languages formally, they study other subjects in both languages.

There are immense psychological and sociological differences between submersion and immersion programmes. Psychologically, students who are extracted from normal classrooms to receive 'special' language instruction are from the very start branded as failures or, at the very best, defectives, who through arduous retraining might possibly be 'corrected' to return to the fold of normal students. The entire programme exists under the negative atmosphere that something is 'wrong' with the students.

On the other hand, immersion programmes begin with 'normal' students, and any progress they make in their second-language learning is rewarded. The emphasis is on positive reinforcement.

A second factor is that teachers in immersion programmes are usually bilingual themselves. Besides the fact that these teachers are not likely to hold their own languages in low esteem, it is important that they understand the types of questions the children are likely to raise. They should also understand some of the linguistic difficulties experienced by the children learning in a second language.

It is not very surprising, therefore, that immersion programmes experience a much higher success rate than submersion programmes. For this reason, the term 'immersion' has taken on positive connotations, and as a result many programmes, especially in the U.S.A., erroneously call themselves

immersion programmes when they are in reality submersion programmes.

It is interesting to note that in most immersion programmes, for instance the French-English ones in Canada, children continue to speak their L1, English, at home and are exposed to the L2, French, only at school. There has been no evidence that speaking English at home deters their progress in French. This finding should be raised when the argument is brought out that minority children may be at a disadvantage because they use a different language at home. The Canadian children did not appear to suffer in French ability from speaking English in the home, and so speaking a different language in the home does not of itself necessarily hinder children's progress in school.

Many studies have been carried out on the relationship between home language and school language. The results are very mixed from language group to language group, from social class to social class, and from age to age. There is even some evidence (Dolson, 1984) that some children who use a mixture of L1 and L2 at home do even *worse* at school than those who speak the L1 exclusively at home. The only conclusions one can draw from the confusing array of results is that if there is any relationship between use of L1 in the home and performance in (L2-medium) school, that relationship is an exceedingly complex one which cannot be described simply by stating that speaking L1 at home is a major *cause* of failure at school. Part of the confusion in experimental results may be due to the attempt to measure language competence as a single skill. If the BICS-CALP duality is taken into account, it might be found that some tests of competence measure BICS skills while others measure CALP skills, with different results.

What, then, can be done for children who speak a language other than the usual language of school? Simply keeping them in the English-medium classroom does not appear to be the answer, for they are at a disadvantage to their English-speaking classmates. Submersion programmes do not appear to work for the reasons outlined above. One possible solution may emerge from the threshold hypothesis, and especially from the developmental interdependence hypothesis. These hypotheses claim that minority children must reach a certain threshold level in their own L1 in order to develop skills in their L2. The suggestion is therefore that minority children be given special instruction in their mother tongue. If Spanish is their first language, then they should

be given not only Spanish classes, but other classes such as mathematics in the Spanish medium, that is, they should not only study their L1 but should use their L1 to study other subjects. Their education would of course not be completely in Spanish; they would naturally attend classes in English with the rest of their English-speaking classmates. The objective would be an additive bilingualism in which they could learn mathematics sometimes in English, sometimes in Spanish.

These are politically sensitive issues. Schools must cope with anti-minority sentiment and the attitude, especially in the U.K. or the U.S.A., that "this is an English-speaking country; let them learn in English". Bilingual education is often seen as special consideration for immigrants who have not earned it and who do not wish to integrate themselves into the fabric of English-speaking society.

The claim of the threshold and interdependence hypotheses is that bilingual education is in fact the *best* way of integrating minority children into English-speaking society. This is because only through the mastery of both BICS and CALP skills in their L1, e.g., Spanish, can they become fully competent in their L2, e.g., English, and only through a mastery of both L1 and L2 can they expect to master other material such as mathematics.

The synergistic effects of bilingual education are difficult for many politicians to understand. Their simplistic reasoning is that if children need to increase their proficiency in English, they should study more English. This view assumes that L1 skills and L2 skills are mutually exclusive and that there is no transfer of linguistic ability from L1 to L2. This view has been challenged, and research now appears to support the view that there is quite a bit of overlap between L1 and L2 skills. This theory is often called the *Common Underlying Proficiency* (CUP) model, as opposed to the traditional view of *Separate Underlying Proficiency* (SUP) model. The sketches in Figure 8.3, taken from Cummins and Swain (1986) represent the two models. Beneath the two sketches is their diagram of how an individual may appear to have two very separate skills, while in reality the skills have much in common below the surface. This model is called the 'dual-iceberg' representation.

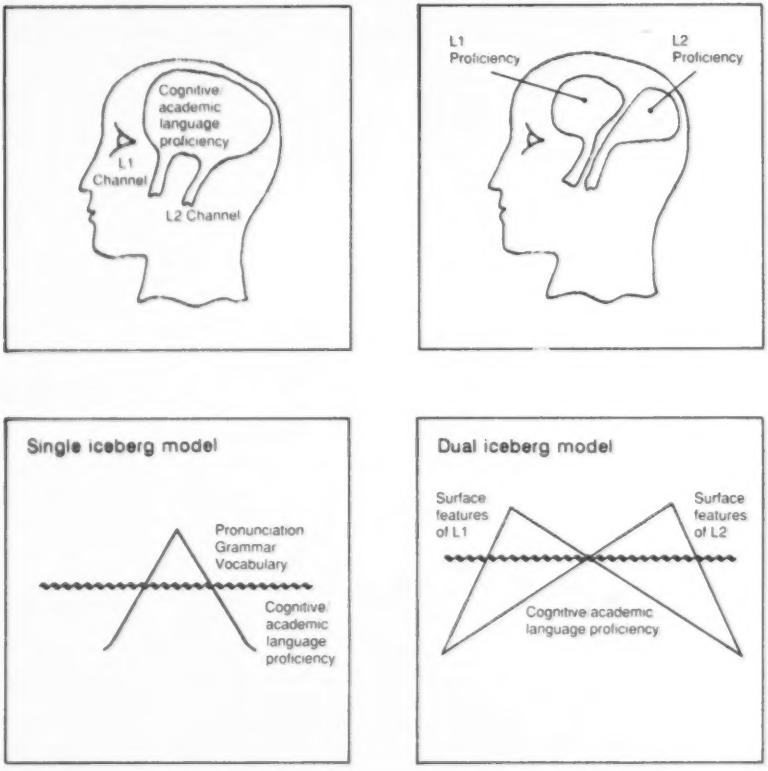

Figure 8.3 CUP and SUP models

The SUP-CUP distinction should remind the reader of the discussion at the beginning of the chapter, on coordinate and compound bilinguals. It will be recalled that coordinate bilinguals, those who develop separate concepts in two languages, appear to realize that the concepts are related or have something in common. This view supports the CUP model. Under the SUP model, a truly coordinate bilingual might have two separate notions for the concept of, say, addition, without realizing that the concepts were related. The dual iceberg representation is a reasonable synthesis of the coordinate-compound dispute, since it retains features of each model: the partial separation of concepts at the surface level, and the considerable common ground below the surface.

The success of many immersion programmes, together with the realization that children can become fully proficient in two languages in the same amount of time that it takes other children to become proficient in one, with no discernible detriment to achievement in other skills such as mathematics, is itself a strong argument in favour of the CUP model. For if the underlying proficiencies in L1 and L2 were truly separate, it should take twice as long to acquire an education in both languages. The apparent transfer of mathematical abilities learned in the L2 but applied to L1 situations is additional evidence that L1 and L2 proficiencies are somehow connected.

It has been stressed above that many of these ideas are not taken seriously by policy makers, who do not grasp the fact that increased proficiency in the L1 may aid L2 learning. The reason that these ideas have become more popular in recent years may be a recognition that many submersion programmes have not worked and that a new line of attack is called for. Gaardner (1970) reviewed the "first seventy-six bilingual education projects" in the U.S.A. and found them woefully inadequate in a report which revealed:

> such inadequate attention - time, resources, and understanding - to the other tongue, as compared to the attention paid to English that, on the whole, the concept of bilingual education represented by these plans of operation seems to be something less than the legislation and its advocates intended. (p. 163)

Much has been written on the defects of bilingual education programmes. To continue this assault is not very productive unless it is possible to cite examples of bilingual programmes which have been successful. One programme in the U.K. which shows promise (Alladina, 1985) is the Secondary Mathematics Individualised Learning Experiment (SMILE). This project, devised by mathematics teachers, attempts to adapt mathematical material to the individual linguistic needs of the pupil. Mathematics is learned through the medium of both L1 (often Bengali) and L2 (English, but often simplified). One emphasis is on the contributions of the various cultures to mathematics. Another major goal is not to "treat second language learners as academically and intellectually inferior but to boost their confidence and self-respect."

In the U.S.A., a programme for Spanish-English bilinguals was introduced in the San Diego, California, City Schools (1982). From kindergarten to grade 3 most instruction was in Spanish, but later, instruction was split equally between Spanish and English. By grade 6, the students had surpassed the national norms of achievement in both languages. Mathematics achievement was also above the national average for grade 6.

In Canada, a programme was set up by the Edmonton, Alberta, Public Schools (1980) for English-Ukranian bilinguals. As in the San Diego project, instruction began in the L1 (here, Ukranian), but later on, half of the academic subjects were taught in Ukranian, half in English. Mathematics was taught in English. By grade 5, the first cohort of bilinguals were performing significantly better in mathematics than normal students.

In both the San Diego and Edmonton projects, it took several years for differences to become apparent. In San Diego, children in the lower grades of the experimental group lagged behind the national norms, while in Edmonton, results were mixed in the lower grades. These findings indicate that bilingual education is not something to be tried for a year or two, and to be abandoned if the immediate results are not satisfactory. It may take several years of instruction before children display the benefits of such programmes.

Developing countries

Most discussions of bilingual education tend to focus on minority languages in the U.K. or the U.S.A., or on the often successful immersion programmes in Canada. Not much is said concerning the developing countries with a colonial heritage of a foreign language. In Africa, for example, the official language of education in most countries is either English or French: the second, third, fourth or fifth language of almost all students. It is no understatement that mathematics education in most African countries is in crisis, for a number of reasons, including lack of funds and poor teacher training. I would like to suggest that the language policy is one of the reasons for the sad state of mathematics education in many of these countries.

In the Ivory Coast, for example, children are taught in French from day one of the first grade, even if they do not know a word of French at the outset. They are taught their mathematics in a language which they do not comprehend, and there has been an

attitude that their native languages are somehow beneath the status of true languages and should be discouraged. At the very best, they become subtractive bilinguals; at worst they become individuals with no true mastery of any one language. In light of the interdependence hypothesis, we can expect that the discouragement of the L1 hampers the development of their L2, French, and the poor development of both, according to the threshold hypothesis, hampers their learning of mathematics.

Other African countries pay more attention to the L1. Children may attend primary school in their own language, but may switch to English or French in secondary school. They may even continue to study their L1 to examination level in secondary school. The difficulty is that language instruction *per se* is always considered distinct from, say, mathematics instruction using a language. In primary school, children may attend English classes while all other subjects are taught in the L1. Suddenly, at the secondary level, they attend L1 classes while all other subjects are taught in English. There needs to be more integration of subject matter and L1.

The results of this problem emerged very clearly from the study of logic in Sesotho and English reported in Chapter 4 (Zepp, 1982). In the first year of secondary school (the first year of mathematical instruction in English), the students performed poorly in both languages. We might take this poor showing as evidence in support for the interdependence hypothesis, since the students had not attained a very high level in either their L1 (Sesotho) or in English. By the fourth year of secondary school, the results of the logic test were significantly higher for the students who took the test in their L2, English. The students were clearly not additive bilinguals. Even though they were attending Sesotho classes, their mathematics was done in English with the attitude that mathematics is an English language topic and that the place of Sesotho is in the Sesotho class, not the mathematics class. One English language specialist (Jones, 1980) suggested to a meeting of mathematics teachers that if students did not understand a concept in English, the teacher might give a brief explanation in Sesotho. She immediately received the opprobrium of the teachers present and gained the notoriety around Southern Africa as being the person who had dared to suggest that the vernacular be introduced into the mathematics classroom. Such a view was unheard of at the time.

Conclusion

Bilingualism is a complex topic. In addition to the existence of many types of bilingualism, there are psychological, sociological and political factors to consider. A programme of bilingual education which succeeds in one setting may fail in another. There are no simple answers. However, the mere fact that many bilingual education programmes have succeeded is cause for research into what factors are at the root of those successes.

Educators and policy makers tend to view the existence of multilingual communities as a problem. It is often seen as a disadvantage to have speakers of many languages in the same classroom. This attitude may be changing slowly, as the cultural and linguistic diversity is being seen as educationally stimulating. This appears to be the case in some multicultural countries such as Singapore, where the richness of the cultural variety is stressed and encouraged.

The are many small ways in which individual teachers can benefit from the different cultural and linguistic backgrounds of their students. For example, the mathematics teacher can use the counting systems of the various languages as a positive teaching aid.

Many of the articles written on bilingual education stress a common idea: the first language of the students must be considered as an asset rather than as a liability. Respect must be shown for the first language as being of equal importance for the child as the (second) language of instruction. This is the true spirit of 'immersion' programmes, despite the proliferation of programmes which pretend to the title.

Experimental evidence on the effect of bilingualism on cognitive development seems to run in similar directions. Truly bilingual immersion programmes where respect is paid to both languages at least do not appear to have any adverse effects on learning in other subjects such as mathematics. They may even have positive effects in such areas as divergent thinking and problem solving. The evidence for increases in overall intelligence is not conclusive, but at least it does not show decreases in intelligence. If such increases exist, they may take several years to develop.

If the threshold hypothesis is correct, as appears to be the case, the threshold levels of proficiency required in order for positive effects to occur may be rather high, since it would be necessary for the child to think and discuss mathematical notions

in both L1 and L2. An educational implication of a high threshold level is that bilingual education programmes should take the language learning seriously and not simply pay lip service to L1 learning. If only a half-hearted effort at L1 instruction is made, the threshold level may not be attained, and the resulting confusion may even produce negative results in children; it would be better not to have given any L1 instruction at all. This has been a real danger in the 1980's, a decade of educational budget cuts in the U.S.A. and in the U.K. Programmes may be cut back to the point where they do more harm than good. It may be that bilingual education programmes are worth doing right or not worth doing at all.

This difficulty applies *a forteriori* to third world countries with limited resources, since they may be tempted to give less-than-adequate attention to vernacular education. But it need not be the case that sound bilingual education is impossible, since most of the teachers in multilingual environments speak the local language anyway. Local teachers might teach partially in the local L1 and partially in the official language, whether English or French. Perhaps the most important thing is a change in attitude away from the superiority of the European language. If third world teachers were more aware of the benefits of L1 proficiency even when applied to L2-medium instruction, perhaps they would pay more attention to L1 teaching as both subject of instruction and medium of instruction.

In particular, developing countries should pay more attention to improving the L1 CALP skills of students. Children probably acquire adequate BISC skills at home, but they need to develop the more academic and formal side of their L1 at school. This can be done by treating the L1 as a separate subject in school, but it can also be achieved by giving some instruction, say in mathematics, in the L1.

In general, then, educators and policy makers in both developed and developing countries must be disabused of the notion that the only step necessary to bring students to mastery in L2-based education is to spend more time studying in L2. But it may be the case that a certain amount of instruction in the L1, both as subject and as medium, may actually be the best way of improving L2 skills. More emphasis needs to be given to L1 instruction as well as instruction of other subjects in the L1.

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CHAPTER 9

THE ALPHABET EFFECT

In this chapter we discuss the effects of written language on logic and on learning. Of course, there are different kinds of written language, notably phonetic writing such as that used in the West, where the Roman alphabet is commonly in use, and pictographic writing, typified by Chinese characters, often called ideographs in the literature. It will be necessary to explore whether differences in learning styles can be traced to differences in the type of written language used.

The extent to which written language affects mathematical thought will have important implications for mathematics teaching:

1. To what extent can mathematics performance be improved through the improvement of reading skills?
2. Can certain learning difficulties in mathematics be traced to deficiencies in reading?
3. Do students who have learned to read a phonetic language have certain advantages in mathematics learning over those who have learned a pictographic language?

The view that the learning of a phonetic language greatly affects the way people think has been stated strongly by Robert Logan (1986) in his popular book, *The Alphabet Effect*. Logan worked closely with Marshall McLuhan in the study of the effect of media on human behaviour, and his writing on the alphabet effect is very much in the spirit of McLuhan's famous phrase, "The medium is the message."

Just what is meant by the alphabet effect? Briefly stated, it is a way of thinking which arises from the use of alphabetic writing systems. Here is Logan's description:

> A person who is literate has a different world view than one who receives information exclusively through oral communication. The alphabet, independent of the spoken languages it transcribes or the information it makes available, has its own intrinsic impacts. These effects have changed the nature of Western civilization so that it differs both from nonliterate cultures and from those societies whose system of writing is nonphonetic, such as the Chinese.

Learning the alphabet, Logan claims, entails the learning of many skills important to the learning of mathematics, in the form of abstraction, analysis, rationality and classification. More specifically, he lists the abilities to

1. Code and decode
2. Convert auditory signals or sounds into visual signs
3. Think deductively
4. Classify information
5. Order words through the process of alphabetization

Firstly, the fact that the letters of the alphabet, unlike pictographic languages, are purely symbolic and have no meaning of their own, means that the learning of such a language also aids in the learning of abstract concepts. This is why, according to the Logan school, abstract mathematical science grew up in the West and not in the East. The Chinese, despite their many technological inventions such as gunpowder, paper and movable type, never developed an abstract theoretical base for their science, and so they were eventually surpassed by Western science-based technology. Abstract mathematical notions such as groups, vector spaces and topologies, could not have come about without the abstracting ability arising from the use of the alphabet.

Secondly, it has been established that the recognition of symbols is essentially a right-brain task, while the processing of sequential information, such as alphabetical letters, is a left-brain task. It has therefore been suggested that cultures which use alphabets will be more highly dependent on left-hemisphere activity than cultures which use pictographic, and hence right-hemisphere oriented written languages.

The implications of this left-right brain distinction are important for mathematics teaching in different cultures. If, for example, a certain mathematical skill is primarily a left-brain

one, then it will be relatively difficult to teach that skill to students who come from a pictograph-based culture. The geometry of spatial relationships, essentially a right-brain activity, will be especially difficult for Western students who have not developed their right hemispheres to the extent that character-based orientals have.

Thirdly, the separation of speech into discrete units of sound such as letters, contributes to an analytical world view in which problems are attacked by dissection into their basic units. We might argue that the Western scientific model of matter, as divided into molecules, atoms and smaller particles, could be traced to the alphabet effect, which at base produces an atomical approach. In this view, the inventions of the real number system and of differential and integral calculus are results of the dissecting ability induced by the alphabet.

Fourthly, since the alphabet has an order, all words can be written in alphabetical order, and this fact gives rise to a propensity to order and classify things. An alphabetically oriented person will tend to view the universe in terms of ordered phenomena. Thus, the biological classification scheme of Linneaus can be traced to the alphabet effect. Mathematicians' preoccupation with order relationships and with the precise definition of sets is essentially a Western, alphabet-oriented tendency.

Logan goes so far as to create a list of comparisons of Eastern and Western cultural patterns. It is useful to this discussion to reproduce that list here, as follows:

EAST	versus	WEST
Ideograms		Alphabet
Right-brain oriented		Left-brain oriented
Non-linear		Linear
Acoustical		Visual
Analogical		Logical
Inductive		Deductive
Concrete		Abstract
Mystical		Causal
Intuitive		Rational
Generalist		Specialist
Decentralized		Centralized
Local		Universal
Integrative		Fragmented

Space oriented	Time oriented
Traditional	Progressive
Cyclic time	Continuous time
Algebraic	Geometric
Concrete science	Abstract science
Order and pattern	Natural law
Craft	Technology
Invention	Exploitation
Technique	System
Customs-mores (Confucius)	Codified law
Harmony of nature (Tao)	Monotheism
Relativity	Absolutism

The above statement of the alphabet effect if at this stage of the discussion only an assertion. We have said nothing so far of the experimental evidence supporting or contradicting the four claims made above. In fact, it is difficult to prove these assertions, for the task is two-fold: we must first demonstrate differences between readers of phonetic languages and readers of pictographic languages, and secondly, one must show that those differences are indeed due to language differences and not to some other cultural difference.

Abstraction

Take, for example, the first assertion, that the use of an alphabet leads to enhanced capabilities in abstraction. In order to support this view, it would be necessary first to show that differences in abstracting abilities exist between, say, Europeans and Chinese, and even if such differences could be established, one would further need to show that among the many cultural differences between Europeans and Chinese, the difference between alphabetic writing and character writing was responsible for that difference in abstracting ability.

Surely we could not assert that the Chinese are incapable of abstractions. Even a colour term such as 'blue' is an abstract one. In this sense, all languages possess abstractions. What Logan probably means by his 'abstraction' is a search for unifying global principles or laws. In this sense, Western monotheism is an example of abstraction, as is the development of a codified system of general, formalized laws which can be applied to

individual situations. In this sense, Logan is no doubt correct that the West has produced more 'abstractions' than the East.

The term 'abstraction' can be confusing, as it can be taken in either of the two senses just described. Other authors (e.g., Hoosain, 1986; Lindsay and Norman, 1977) prefer to describe the cognitive styles of cultures as being either 'top-down' or 'bottom-up'. Top-down cultures are typified by the use of general laws applied to specific tasks, while bottom-up cultures tend to build up solutions from smaller perceptions. Hoosain describes the difference as follows:

> The former [bottom-up] are initiated and largely determined by perceptual data, but the latter are dominated by knowledge of the possible interpretation and conceptualization of sensory experiences. For example, in reading a word, factors like the size and clarity of the print affect bottom-up processes. Knowledge of the language possessed by the reader, on the other hand, affects top-down processes. (p. 39)

If we consider the list of East-West cultural differences cited above, we can see at once that many of the differences can be viewed as top-down versus bottom-up differences. Codified law, monotheism, system, natural law, abstract science, universal, and absolutism are all words which describe top-down processes.

But what about 'alphabet'? Is the transcription of sounds into abstract symbols a top-down process? Building up words and meaning from letters may sound more like a bottom-up process. Putting together the (by themselves) meaningless symbols to form meaningful words could be construed as a microprocess rather that a macroprocess. The top-down element comes from the reduction of all human speech to some 26 elements. The alphabet becomes a general rule for representing speech.

The important point, however, is that the use of the alphabet, along with almost any other phenomenon, contains both top-down and bottom-up aspects. Reading a book, for example, contains the bottom-up process of being able to recognize the letters of the print (which may be small or badly written), as well as the top-down process of assigning meaning to the words.

Recognition of the fact that all tasks have top-down as well as bottom-up elements makes Logan's argument sound rather over-simplified. The argument that the alphabet is at the root of the Western tendency towards top-down processes loses some of

its force when we consider that the use of the alphabet has many bottom-up elements as well.

No discussion of Eastern versus Western scientific thought would be complete without reference to the most distinguished scholar in the field, Joseph Needham (1978), whose six-volume opus, *Science and Civilization in China*, remains the definitive work on the subject. Needham concurs with the first part of Logan's argument: that there are indeed profound differences in scientific thought between East and West. But he is reluctant to ascribe those differences to the differences in written languages.

Here is what Needham says about Chinese mathematics:

> There were, then, fundamental differences between what may be called the dominant attitudes of the Chinese and Western civilisations. In contrast with the Greek genius for geometry, the Chinese had a gift for arithmetic and algebra. Now there is something suspiciously similar between the abstractness of geometry and the abstractness of Roman law. But for Chinese law such an abstractness is inconceivable. Just as Greek geometry dealt with pure and abstract figures, the size of which was quite immaterial once the principle had been accepted, so Roman law dealt with codified abstractions. But the Chinese preferred to think only on concrete numbers, and, similarly, of concrete social circumstances when it came to matters of law. (p. 304)

Thus, Needham is very much in agreement with Logan on the point that the Chinese are less 'abstract' than the Greeks in many areas of thought. He sees the tendency towards bottom-up processes in Chinese thought, and the fact that those processes appear in law, science, philosophy and government, leads him to conclude (in very top-down fashion, we might add), that there must be some underlying and unifying reasons for those tendencies. Where he differs from Logan is in his assessment of the reasons for those differences:

> The reasons were, first, that the Chinese acquired a great distaste for precisely formulated codified law, a distaste that arose from their unhappy experiences with the Legalists during the transition period from feudalism to bureaucracy. The element of natural law became

> relatively more important in China than in Europe. But the fact that so little of this natural law was expressed in formal legal terms, and also that it was overwhelmingly social and ethical, made any extension of its sphere of influence to non-human Nature impossible. Thirdly, the idea of a supreme being, though certainly present from the earliest times, soon lost all personal and creative qualities. As a result, the development of the idea of precisely formulated abstract rational laws which could be deciphered and re-stated simply because there had been a rational Author of Nature did not occur. (p. 306)

Not only does Needham's summary not even mention the alphabet effect, it provides, between the lines, some powerful arguments against it. Firstly, the Legalist period showed that the Chinese could and did codify the law, but abandoned it for social reasons. Secondly, the Chinese could and did develop natural law, but abandoned it due to their experiences with codified law. Thirdly, the Chinese could and did have a monotheistic religion, but similarly abandoned it.

In the course of his six tomes Needham naturally treats the notion that the Chinese language may have some effect on its science. His view is the following:

> There is a commonly received idea that the ideographic language has a powerful inhibiting factor to the development of modern science in China. We believe, however, that this influence is generally grossly overrated. It has proved possible in the course of our work to draw up large glossaries of technical terms used in ancient and medieval times for all kinds of things and ideas in science and its applications. (p. 38)

Logan points out correctly that the invention of words for abstract concepts is only a small part of his notion of 'abstraction', which we have called 'top-down' processing. He goes further to show that many, if not all, of the Chinese abstract terms are compositions of characters representing concrete items. He fails to provide examples, but at least one can be presented here to illustrate Logan's argument. The Chinese expression for algebra is *dai shu*, a combination of the characters for 'represent' and 'number'. It certainly does not sound very 'scientific' to Western

ears to hear the abstract term 'algebra' being given the concrete equivalent 'represent number'.

But most Western scientific terms are put together in the same way as the Chinese ones. They only *sound* abstract because their concrete roots are Greek or Latin. The word 'quadrilateral' has an abstract ring to it only because it is Latin, but the Latin root is very concrete and simply means 'four sides', just as the Chinese equivalent, *si bian xing* means 'four-sided shape'. The English 'polygon' is a composition of the concrete Greek words for 'many angles', just as the Chinese equivalent is either *duo bian xing* ('many-sided shape') or *duo jiao xing* ('many-angled shape').

The argument that Chinese does not express abstractions is not tenable. We may sum up the discussion on the Western tendency to abstraction by stating that there is widespread agreement on differences between East and West in the area of top-down versus bottom-up processes. But to ascribe those differences to differences in writing systems remains more a speculative belief than an experimentally demonstrated phenomenon.

Right brain versus left brain

It has been shown fairly conclusively through experimentation that readers of alphabetic languages read the words largely using the left hemisphere of the brain, while readers of pictographic languages perceive their characters with the right hemisphere. The experiments usually take one of two forms - either studies done with normal subjects using a tachistoscope which flashes stimuli to subjects in order to record their response times, or else studies on split-brain or otherwise brain-damaged subjects.

In the tachistoscope tests, stimuli such as words are flashed to subjects in either their left or right visual field, and the time it takes the subjects to respond is measured. Since the right hemisphere controls the left visual field, a fast response to a left visual field stimulus is taken as a sign of a right-brain dominance in the perception of the stimulus. The most common test of this sort is called the Stroop test.

In particular, when Westerners (alphabet readers) are shown words, their Stroop response times are generally faster when the words are flashed to the right visual field, indicating

left-brain dominance. On the other hand, when Chinese subjects are shown characters, they display a right-brain advantage. This is a marked and very demonstrable effect.

The typical data from brain-damaged subjects arises from the relative impairment of one area of the brain. The following is a description of such an experiment (*Beijing Review*, 1987):

> Experts found that a wound in the temporal bone can destroy the written language capacity of Caucasians. A similar wound has only a minor effect on the capacity of Japanese people and no effect at all on the Chinese. However, a wound in the parietal bone does not reduce the ability of Caucasians to write and understand written language but can seriously impair the Japanese. Chinese patients with such a wound completely lose their ability to cope with written language. (p. 26)

The obvious explanation for these reported effects is that they are due to differences in language - Caucasians, who read a phonetic language, must use different parts of their brain than Chinese, who read a pictorial language. The partial impairment of the Japanese lends support to this explanation, because the Japanese written language is a combination of phonetic symbols (called *kana*) and borrowed Chinese pictographs (called *kanji*). If Japanese reading is part phonetic and part pictographic, then damage to a specific brain area should be expected to result in only partial impairment of reading ability.

These results for Japanese subjects are borne out by a well-known study by Sasanuma and Monoi (1975), who found that injuries to Broca's and Wernicke's areas in the brain lead to deficiencies in *kana* processing but not in *kanji* processing. Deficiencies in *kanji* processing alone arise if lesions leave these areas intact, but cut them off from the surrounding cortex. The claim is, of course, that *kana* and *kanji* characters are processed in different ways, and this claim naturally supports the hypothesis that phonetic writing is processed differently from pictographic writing.

The studies cited so far seem to paint a clear distinction between the processing of pictographic writing and phonetic writing. Subsequent research, however, has blurred this distinction considerably. One complicating factor is that no task is 100% right-brained or left-brained. All tasks involve a combination of the two. Or perhaps it would be better to say that

in many respects the brain acts as a whole and should not be considered merely as a set of isolated regions of activity.

A second difficulty important for mathematics education is that hemispheric dominance may not reflect hemispheric capacity. In other words, even if a student tends towards more right-brain activity, he or she may still be quite talented in using the left brain.

The simple experiments described above provide evidence that Chinese readers perceive individual characters with a marked right-brain dominance. But that is no proof that they display that same tendency when actually reading text for meaning. Tzeng (1979) showed that when shown meaningful pairs of characters, Chinese subjects displayed a *left*-brain advantage in identifying the pairs. Moreover, when the Chinese subjects were shown random pairs of characters and were asked to identify the ones which had a meaning in Chinese, they again displayed a marked left-brain dominance. Thus the higher order reading skills, even when applied to pictographic languages, may reflect a universal left-brain dominance.

Experiments with brain-damaged patients have led to similar results: Guo (1984) examined aphasia patients who had suffered damage to one hemisphere of the brain. The results are given in Table 9.1.

Table 9.1 Probability of aphasia as a result of damage

Handedness		*Left-handed*		*Right-handed*	
Location of damage		left hem.	right hem.	left hem.	right hem.
Probability of Aphasia	Chinese	0.83	0.33	0.75	0.19
	Western	0.65	0.48	0.59	0.03

If we focus on the normal right-handed subjects in Table 9.1, we see that a majority of both Chinese and Westerners lose their language ability when the left hemisphere is damaged, but that the Chinese appear even *more* dependent that the Westerners (0.75 as compared to 0.59) on the use of their left brain in language ability.

Thus it appears that although the Chinese may recognize individual characters with a right-brain dominance, they actually

read in much the same way as Westerners. In fact, Chinese script as code for discourse is not as different from alphabetic script as might be expected.

Firstly, the Chinese language has a propensity for using pairs of characters to form 'words', whose meaning may seem to have little in common with the meanings of the characters taken in isolation. For example, the Chinese *dong xi*, meaning 'thing', is composed of the characters for east and west. The Chinese read the character pair as a meaningful unit, and surely do not pause to consider that the pair means 'east west'.

Secondly, the Chinese language is not really a pictographic language, in that the characters are not actually recognizable pictures of objects. A small percentage of the simpler characters are derived from pictographs, but over the centuries have become so stylized that they now hardly resemble the original pictographs. Certainly the average Chinese reader does not mentally process each character as a picture. For this reason, most sources today refer to Chinese characters as 'ideographs' rather than pictographs.

Thirdly, the Chinese written language is much more phonetic than many people realize. Most characters are composed of two parts, a phonetic radical, which tells how the character is pronounced, together with a category radical, which gives a clue to the meaning. For example, the following characters are all pronounced *gou* because of the identical right-hand portion:

沟　　钩　　构

The first ideograph has the 'water' radical on the left, which indicates that it is the *gou* which has something to do with water. A Chinese speaker will recognize that it must be the *gou* which means 'ditch' or 'channel'. The middle character has the 'metal' radical on the left, which indicates that it is the *gou* which has something to do with metal, namely a hook. The last character has the 'wood' radical on the left, and can thereby be deciphered by a Chinese speaker as the *gou* which means 'construct'.

We are now in a position to summarize the findings on left- and right-brain differences. There are clear differences between readers of characters and those of letters in the way in which they identify language symbols. But when it comes to higher order

skills such as reading and recognizing meaning, clearcut differences have not been found. There is little evidence, if any, that mathematics learning is affected by left- or right-brain dominance due to the nature of the written language.

In fact, the entire question of hemispherical dominance may be becoming less and less fashionable, as the experimental evidence increasingly indicates that the brain is much more complex than the simple model that the left brain performs certain tasks while the right brain performs other tasks. For example, Ojemann (*Newsweek*, 1983) found a total of 20 different sites on the *right* side of the brain that affected mathematics students' ability to multiply and divide.

The attitude that the answers are not so simple has been expressed by one of the foremost neuroscientists in the field, Jerre Levy, in an interview with *Omni* magazine (1985):

> Reality has really humbled me. I had believed that reality was going to be beautifully simple and that if you thought up something that seemed simple and elegant, it had to be true. Well, now I've changed my mind. The reality of the human brain, I've decided, is contentious, misleading, complex, and flirtatious.
>
> One of the ideas I had was that the right hemisphere of the human brain attends to and represents the holistic, configurational aspects of its experience and that the left hemisphere is more analytic. The right hemisphere sythesizes things into global form, whereas the left seems to pay attention to specific and detailed features. This is now a popular notion, but I think it may be wrong. (p. 97)

The same view has been expressed more bluntly by D. Weinberger of the American Institute of Mental Health (*Newsweek*, 1983):

> A lot of old theories about right brain and left brain are nonsense. Things are not as localized as we thought. (p. 39)

In fact, if we examine more closely Logan's list of Eastern versus Western characteristics, we notice some discrepancies from the simplistic left-brain versus right-brain model. Orientals, according to the list, are better at algebra, while

Occidentals excel at geometry. If the hemispheric dominance theory were true, would we not expect the right-brained Orientals to excel at geometry, since the right-brain is more spatially oriented? The analytical skills of algebra should be more left-brained and hence should be expected to appear on the list of skills of Occidentals. Moreover, if the right hemisphere is more synthetic and holistic, we might expect the Orientals to be more concerned with general rules, codified laws, and monotheism.

Analytical ability

One of the arguments of the proponents of the alphabet effect is that alphabetic words must be broken down into their component letters, thus promoting the analytical skills of dissection and scrutiny. The application of this argument to mathematics learning is that readers of alphabetic writing may be better able to analyze a mathematical concept or problem by scrutinizing its various facets.

From this discussion, we should not necessarily expect to find differences in analytical skills between Occidentals and Orientals due to their written languages, for the Chinese and Japanese languages have strong phonetic components. Indeed, the dissection of Chinese characters into phonetic radicals and category radicals gives rise to the same sort of analysis as with alphabetic words.

The issue is further blurred by the fact that reading of English words is not purely phonetic. Rapid readers surely do not waste time sounding out each word by dissecting it into its componenet letters; rather, words are recognized as meaningful units. Furthermore, there is no one-to-one correspondence between sounds and letters, as evidenced by long and short vowels, silent letters, or hard and soft c's or g's. Most English words cannot be pronounced simply by breaking them down into letters.

Another difficulty in assessing the situation is that in many countries where ideographic writing is used, Romanizations also exist. For example, in mainland China schoolchildren also learn to read the *pin yin* alphabetization of their language. A similar situation prevails in Taiwan, where schoolchildren learn the *zhuyin fuhao* phonetic system before studying characters. The Japanese and Korean languages are also partly phonetic.

These difficulties do not refute the claims of analytical superiority for readers of alphabetic languages. They merely

render experimentation difficult. It is virtually impossible experimentally to isolate any possible effects due to alphabetic writing.

One argument, however, may be proposed: the analytical method depends on a distance viewpoint which must treat data as neutral. In the Western world view, words have no reality in themselves. For example, a Western reader knows clearly that the word 'apple' is not really an apple and cannot be eaten. In many non-Western cultures, however, the distinction between word and reality is much less clear. In several African societies, for example, we may observe traditional medicine men writing a word such as 'health' on a blackboard, shaking the chalk dust from that word into a container, and then feeding that dust to a patient as medicine. Similarly, magic words are often written on paper, and the paper swallowed as medicine. In the Orient, words concerning death or poor health are quite taboo, for speaking such words may actually bring death or poor health to the speaker. In Taiwan, no car may have the license number 884, because the pronunciation of the number sounds like the pronunciation of 'Papa is dead'. Many hotels in Taiwan do not have a fourth floor, because the Chinese pronunciation of the number 4 sounds like the word for death.

A few controlled experiments have been carried out which examine this effect. In an interesting and common version of the Stroop test, subjects are flashed colour words written in colours. For instance, the word 'blue' may be written in green. Subjects are asked to read the word as quickly as possible. Among almost all subjects tested, in whatever language, subjects will identify a colour word written in its own colour more quickly than a word written in a different colour. For instance, subjects will take longer to read the 'blue' if it is written in green than if it is written in blue. Relevant to this discussion is an experiment by Biedermann and Tsao (1979), in which Chinese showed a significantly greater Stroop interference on the test than Westerners did. Subjects were shown colour words printed in a different colour (termed 'print'), a sample patch of that same colour ('patch'), the colour words written in the same colour ('colored'), and the colour words written in black ('black'). Table 9.2 compares Chinese and Western responses to the test.

Table 9.2 Stroop test by Chinese and Westerners

	PRINT (conflict)	PATCH (control)	INTERFERENCE (print-patch)	COLOURED (same)	BLACK
English	872	471	401	425	349
Chinese	1269	672	597	496	406

Mean Stroop times (milliseconds)

The significantly higher interference for Chinese is an indication that there may be heightened interference between the word itself and the meaning of the word. If the Chinese expect the reality of the word to lie within the written symbol, they may experience more confusion when the character for 'blue' is written in green ink.

Further support comes from another variant of the Stroop experiment with Japanese students, whose language contains both phonetic (*kana*) and ideographic (*kanji*) characters. Morikawa (1986) showed that Stroop interference was greater for *kanji* characters than for *kana* characters. This finding supports the view that ideographic characters are more meaningful in their own right than phonetic ones are.

Such evidence is only circumstantial and does not prove that non-phonetic languages lead to confusion between word and reality, thereby hindering analytical thought. Alphabetic words are not inherently exempt from confusion with reality. Still, it seems reasonable to suspect that alphabetic words, which are simply sets of phonetic letters, are difficult to be taken as symbols in themselves. Chinese ideographs, many of which contain an element of meaning in the form of a category radical, are far more likely candidates for symbols. We would not be surprised to see Chinese people in a temple praying to the character for 'Buddha', but it would be rare to see a Western Christian praying to an altar containing only the alphabetic word 'Jesus'.

But even if we admit that phonetic languages are more likely to be neutral, it is still a long jump to the conclusion that the neutrality of words leads to an increased analytical ability. There is little evidence, if any, that Western students are superior to Orientals in the analytical skill involved in mathematics

learning. Even if such a superiority could be documented, it would be quite difficult to link it to the neutrality of phonetic language.

Classification

Phonetic language facilitates alphabetical classification, and Western science has provided many schemes. Western science is based on the concept of a well-defined set, such that any object under consideration may be said to be either in the set or out of the set. Is there a link between phonetic language and Western mathematics, as the proponents of the alphabet effect claim, and if so, is the link a causal one?

As we saw before, it is exceedingly difficult to provide experimental evidence for any such link. We must first establish that Westerners are more inclined to categorize than are speakers of pictographic languages (assuming we could find a totally pictographic language). Even the Chinese, whose language is less phonetic than the Western alphabet, have in the past been avid categorizers. As Needham (1978) has pointed out, during the Chinese Legalist period, laws were codified, and matter was classified according to the five elements of metal, water, earth, wood and fire, as well as according to the ten heavenly stems and twelve earthly branches. Even today, there is a fixed order of 226 radicals, listed partially in the order of the number of pen strokes necessary for drawing them, and this order is uniformly used in looking up Chinese characters in the dictionary.

People from all cultures and linguistic backgrounds categorize objects. Some psychologists assert that the tendency to categorize is a universal human trait, and precedes language. Such is the view of Lenneberg (1967):

> Thus categorization by a principle, or the formation of an (abstract) concept is apparently prior to and more primitive than the association of a sound pattern with a specific sensory experience.

Any discussion of the relative *amount* of classification done by various cultures is therefore not fruitful. The arena for debate must be the *nature* of the classification schemes. The real question is whether readers of alphabetic languages form special kinds of classification schemes. We have discussed the effect of language on

concept in Chapter 3. Here we look at the effects arising from reading alphabetical words.

Firstly, it has been pointed out that little direct evidence can be produced on this subject. The only direct arguments would be observations of the kinds of classification systems used in the West as compared to those used in Eastern and in pre-literate cultures. However, most of the arguments proposed by exponents of the alphabet effect are of an indirect nature. Their claim is that the relatively small number of letters in the alphabet made printing with movable type a workable proposition. The rapid dissemination of knowledge which resulted from the widespread use of the printing press rendered the classification of knowledge both simple and necessary.

True, the Chinese can claim to have invented movable type long before Gutenberg, but the large number of Chinese characters made its impracticable for widespread use. Even into the twentieth century, most documents in China were copied by hand.

Secondly, the social revolution in Europe around the time of Gutenberg led to the sharing of knowledge among an ever-growing percentage of the population, and so the printing press became indispensable in providing information for the masses. In the East, on the other hand, knowledge was still the property of a very small elite. The need did not arise for large numbers of copies of documents.

The argument, then, is not principally that the alphabet had any direct, deep-seated effect on the classifying powers of the mind; rather, the use of the printing press (itself facilitated by alphabetic writing) and the concomitant dissemination of information to large segments of the population, induced a mentality of information technology, that is, a preoccupation with the communication of information. From Gutenberg's time onward, scientific problems could not be dissociated from the problem of dissemination. Finding the solution is not enough; the solution must be communicated. This was not the case in the Orient. The concept of storing information in databases was therefore much more prevalent and necessary in the West.

In pre-literate societies, knowledge was considered the domain of only a few specialized individuals, for example the witch doctor, the story-teller or the African *griot*. The repository of knowledge was in a single person. The idea of freedom of information is a very modern one, and has at its roots the printing of typed alphabet. If information is to be free, it must be stored

and readily accessible, and that necessity leads to a preoccupation with classification schemes, hierarchies, relationships and order.

Pre-modern societies may have been interested in the order of the universe, and they may have invented elaborate taxonomies. But it is only in modern times, with the advent of mass communication, that man has wished to *impose* on information the order which best suits the storage and communication of that information. The Aristotelian essentialist theory that a natural classification exists in reality and that man must simply find it is similar to the Taoist school of the Orient, but that view has given way in the West to a 'polythetic' view that the classification system is a choice of the *classifier* rather than a fixed property of the classified.

It is this attitude of arbitrariness that gives Western mathematics much of its impetus. Western mathematicians can invent arbitrary classifications, such as the set of all numbers with some interesting property, and they may proceed to analyze what other properties those numbers may have, or in terms of our discussion, what other properties may be useful in storing or communicating that set of numbers. This attitude is quite the contrary to that of the ancient Pythagoreans, who believed that properties of numbers existed as part of the harmony of the universe.

When choosing mathematics curricula in various societies, it is necessary to realize that the organization of data is more important to some societies than to others. Matrices, which are primarily arrays of data, may be of little interest to some societies. But this line of thought is moving away from the alphabet. We shall take it up again in Chapter 11 when we discuss cultural differences.

Summary

On most counts, the alphabet effect does not stand up to experimental scrutiny. For one thing, it is virtually impossible to compare alphabet readers to pictograph readers, since it is so difficult to find pure pictograph readers. If we could examine the psychology of the ancient Egyptians, who used true hieroglyphics, we might be able to draw some conclusions. Indeed, the Egyptians had to develop complex mathematics to build their pyramids, and so it seems that their hieroglyphic writing was not a drawback to the development of their geometry. Many studies have attempted to

compare Orientals with Occidentals, but this comparison is not usually appropriate, since Oriental writing is not completely pictographic. It contains many of the phonetic and analytical elements of the alphabet.

The other major difficulty with research on the alphabet effect is that any possible alphabet effect cannot be separated from other cultural variables. You simply cannot hold all variables except alphabetic writing constant in an experiment. Any evidence will be necessarily circumstantial.

I have perhaps been unduly harsh on Logan and the advocates of the alphabet effect. In their defence, it should be said that they are well aware that their evidence is circumstantial. Logan's book is full of disclaimers to the effect that he is not claiming a direct causal relationship between alphabet and thought. He admits on almost every page that his evidence is circumstantial, and constantly uses phrases such as, "it is probably no accident that" or "could it be just a coincidence that". His arguments often seem to suggest a causal link, but he manages to wriggle off the hook by claiming that the connection is not one of cause and effect.

But if the acquisition of alphabetic writing does not actually cause changes in cognitive behaviour, what other kind of relationship could there be? Could it be, for example, that the alphabet is actually the effect and not the cause? That certain culturally instilled ways of thinking led the West to invent and use the alphabet? Perhaps an ingrained bottom-up approach to life was the *cause* of Chinese ideographic writing, rather than the effect.

In the absence of further discussion of the actual nature of the alphabet effect, it loses some of its force in applications, especially in mathematics education. For even if Western mathematics can be linked to the alphabet, a non-causal link destroys any grounds for improving mathematics performance by improving certain alphabetic reading skills. We cannot claim that Chinese students, for example, will benefit in mathematical ability by learning to read English.

We are very near the point of rejecting the alphabet effect as either false, unprovable, or useless. Is there anything that can be salvaged?

One demonstrable effect which may be of some use is the difference in Stroop interference between character readers and alphabet readers. The treatment of symbols as having a reality of their own (the non-Western tendency) may have consequences for mathematics education. Would it help Western students, for

example, if some mathematical symbols contained intrinsic clues to their meaning? this is already done to some extent when we say, "let *d* be the distance..." For the letter *d*, standing for a word beginning with *d*, in this instance gives the reader a clue to the physical quantity involved in a more meaningful way than would the letter *x*. Perhaps mathematics learning could be improved by letting more symbols have similar clues.

Still, there is no evidence that the Stroop effect has any effect on mathematics performance. It is even possible that making mathematical symbols more iconic (or otherwise meaningful) would lead away from the abstraction eventually desirable in mathematics.

There is also some evidence, as we have seen, that certain learning disorders can disrupt alphabetic reading ability without affecting ideographic reading ability. The experiments done with brain-damaged students lead in this direction. An often quoted study by Rozin (1971) showed that American children with certain severe learning problems in reading could still be taught to read basic Chinese characters. The finding could have implications for the teaching of other subjects such as mathematics to children with learning disorders.

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CHAPTER 10

SOCIOLINGUISTICS IN THE CLASSROOM

Much of the material in this book and elsewhere deals with those elements commonly associated with language: vocabulary, grammar, word order, etc. But contemporary theories of language in its full sociolinguistic sense see words and phrases as only a part of the communicative process. Communication also involves tone, body language, and a host of other techniques quite apart from the actual words used. Quite often the way a sentence is spoken conveys more meaning than the sentence itself, and quite often what is not said conveys more meaning than what is said.

This chapter will treat the message which teachers send to students, sometimes quite distinct from the mathematics lesson itself. These meanings may deal with social class or racial attitudes, or may tip students off as to what the teacher *really* considers important. The question must then be raised whether these more or less hidden messages are received in different ways by members of different races or social classes. One of the objectives of this chapter is to alert teachers to the fact that they are teaching values, attitudes, and prejudices towards situations both inside and outside the classroom.

The correctness of language

Mathematics teachers may insist that their students speak and write correctly. If a child states "there ain't no such number," the teacher may be tempted to restate the phrase in correct English as "there isn't any such number." This is a clear example of how teachers are teaching something besides just mathematics in their classrooms. It is useful to examine the motivation for insisting on correct language, and in so doing, a clearer picture may emerge as to just what 'correct' language is.

It will be clear from the sociolinguists' definition of language ability as 'communicative competence' that they are interested less in the 'correctness' of language than in how well it serves to communicate. One sentence (including the way in which it is spoken) is better than another only if if conveys its message more clearly and not because it is more 'correct' than another. What is correct or appropriate language in one situation may be inappropriate in another. Is it not true that "there ain't no such number" communicates its message quite clearly? If so, then what is wrong with it?

The teacher who corrects the child's English may argue that the English language is a work of high art which has reached the pinnacle of perfection after centuries of development. As a result, the use of 'sloppy' English is a dangerous attack on the language itself. If everyone spoke sloppily, English would lose its power, its precision, and its beauty.

Suppose, then, that the child had answered the teacher in 'correct' Chaucerian English, or in Elizabethan English, or even in Victorian English. The teacher's response would probably be that such language is old-fashioned and not proper modern English. In fact, some words fashionable only one or two years ago may now be considered out of date. Clearly languages change, and quite rapidly at that. How does one define correct up-to-date English if it may be incorrect tomorrow?

No doubt the best rejoinders for the defender of the King's English are:

a) Language can change for the better as well as for the worse. Perhaps some criteria could be established for what changes are acceptable. For example, language which follows the orderly rules of syntax is acceptable, while a phrase such as "he are" is not. Such criteria would better distinguish 'sloppy' English as English which violates the apparent logic of the language's structure.

b) While change may be inevitable, it should be retarded and kept to a minimum. We can read books written over a century ago because the language has not been permitted to change very much. It would be a pity if language were permitted to change so rapidly that we could not even read last year's newspapers.

In response to argument a), it is possible to cite numerous inconsistencies even in correct English syntax. A phrase such as "I'm doing it, aren't I" is perfectly acceptable in most circles, and even "it's me" enjoys wide acceptance. Should we not classify as

'sloppy' such anomalies as 'public school', which in British English actually means 'private school'?

In response to argument b), one may point out that the word 'ain't' has been in common use for centuries. A reader would have no trouble understanding it in a Dickens novel. In fact, if we were to eradicate all use of the word 'ain't' we could no longer properly read Dickens. There should be no question that 'ain't' is very much a part of the English language and that its eradication would detract from the richness of English. Its use in a Dickens novel conveys a nuance of meaning that could not be conveyed otherwise.

One begins to suspect that 'ain't' is frowned upon for reasons other than the reasons usually stated. For otherwise, 'sloppy' terms such as 'public school' and 'aren't I' would be equally frowned upon. The reason is probably that 'ain't' tags its speaker as belonging to a socially undesirable class. When the teacher corrects the child, she is conveying the message that he should learn to speak language appropriate to the more educated classes. That message is part of the the more general message that education's goal is to raise the social status of its beneficiaries. That is, of course, the message of "My Fair Lady". It is almost possible to equate 'correct' English with 'upper class' English.

The teacher who corrects the child's 'ain't' is certainly not consciously criticising the child for his or her class background, although the child may read it that way. The teacher no doubt realises that in the competitive business world, the person who says 'ain't' is not likely to get the job, and the child must be made aware of this fact. Thus, the teacher in good faith corrects the child in order to help the child succeed in life.

Thus, there is nothing inherently wrong in correcting students' English, as long as teachers are well aware that no language is correct in an absolute sense. Language may or may not be appropriate to a situation, such as a job interview, but to state that one type of language is somehow inherently better than another promotes class or ethnic antagonism.

Language of the rich and the poor

Insistence on 'correct' English is only one example of how students from poorer families are placed at a disadvantage in schools. In the past two decades, much research has been carried out on other differences between rich children and poor children. One of the major differences is the importance of goal-seeking.

Children from lower class homes are usually not nearly so goal-oriented as middle class children. This fact has been demonstrated strikingly in research into family environments, for example, by Schoggen (1969), who attempted to record how often children were directed by others (e.g. parents) to seek a goal. As expected, most lower-class children received very few instances of goal direction as compared with middle class children.

Research has shown that goal-oriented language displays certain characteristics. An early study by Moffett (1968) examined the language of children in different situations. He found that if children talk about themselves, especially when they are in a goal-seeking situation, they will use much more complex language. This finding applied to upper and lower class children alike.

A second early study by Strandberg and Griffith (1968) asked young children to describe photographs which they had taken themselves. In some cases they were asked to phtotgraph only things which interested them personally, while in other cases their choice of photograph was dictated by the experimenter. The types of language used in the two types of descriptions are interesting to compare. The following is a five-year-old's description of a horse which the experimenter had called for:

> That's a horse. You can ride it. I don't know any more about it. It's brown, black, and red. I don't know my story about the horse.

Compare those short, simple sentences with the language used by the very same child when describing a self-chosen picture of the child's treehouse:

> There's a picture of my tree that I climb. There's where it grows and there's where I climb up - and sit up there - down there - and that's where I look out at. First I get on this one and then I get on that one. And then I put my foot under that big branch that are so strong. And then I pull my face up and when I get a hold of a branch up at that place - and then I look around.

This is clearly goal-oriented language (the goal of climbing the tree), and is considerably more complex than the description of the horse, despite a few grammatical inconsistencies.

The point is that middle and upper class children who are brought up in goal-oriented environments are more likely to use and understand complex language than are their working class schoolmates. This difference poses a dilemma for even the mathematics teacher. Should explanations of mathematical terms be given in as simplified language as possible, or should more complex language be used on grounds that the students is going to have to learn that sort of language eventually? This is almost the same argument as the one described above, over whether or not to correct a child's English. If a child wishes to apply for a job, go on for higher education, in short, to become a member of the educated class, he or she must master the complex language required.

There is of course no straightforward arswer to this dilemma. The least one can say is that teachers must try to steer a reasonable course between language which is too complex to be understood at all and oversimplified language which would convey a bad impression to potential employers and the like.

Questions versus directions

In studies of family environments similar to those described above, lower class parents were found to tend to speak to their children in the form of commands, directions, and behaviour-requiring speech, while middle and upper class parents asked more questions, tried to get the children to seek their own solutions to problems, and used more positive reinforcement in shaping their children's behaviour. Typical of such studies is that of Henderson (1972), who found that, in comparison with working-class mothers, middle class mothers used more abstract definitions, information which pointed to the use of strategies by children, and vocabulary relating to emotions or feelings. A study by Cook (1971) produced results along the asme lines, also noting the tendency of lower-class mothers to use commands with no rationale as to the outcome of the child's actions. The lower-class mothers tended to argue from a position of authority, as evidenced by such expressions as "You do that because I tell you to", or "I'm your mother, that's why". Outside authorities such as policemen are also cited by working-class mothers, according to Newson and Newson (1967). On the other hand, middle class mothers are far more likely to support their requests with reasons, and to state why a child should do something, as opposed to "because I say so".

The implications for such research in the classroom are that lower class children are used to a certain kind of language at home, and are not accustomed to hearing the language of a different social class in school. They respond more to threats of authority than to cause-and-effect explanations. This is the conclusion of the Henderson study cited above:

> [For the middle class child] there is little discontinuity between the symbolic orders of the school and those to which he has been socialized through his family. Whereas for the working-class child there is a hiatus between the symbolic orders of the school and those of the family. The genesis of educational failure, according to our findings, may well be found in the pattern of communication and control which are realizations and thus transmitters of specific subcultures. (p. 329)

The work of Basil Bernstein

These ideas of the effects of social class and language have received perhaps the most well-known formulation in the early writing of Basil Bernstein (1961). Bernstein saw a clear relationship between the type of communication used by children and their social class, and observed that individuals learn their social roles through language. He saw the relationship as a circular one, wherein one social class determines what is considered 'correct' language, but where that same language tends to reinforce social class. His theory is most well-known for its division of language into two types: *elaborated code* and *restricted code*. Elaborated code uses complex sentence structure, grammatical order, prepositions to show logical and sequential ordering, and a high degree of hierarchical organization. Restricted code uses shorter sentences and is much more limited in its choice of adjectives, adverbs, and prepositions. Now in familiar settings, people of all classes use some sort of restricted code. Everyday language at home and among friends is full of short, simple phrases which are certainly restriced code. However, according to Bernstein, lower-class children have little access to the elaborated code, which is the code used in school. As a result, these children suffer deprivation in school and quickly fall behind their middle class schoolmates, who are more accustomed to working with the elaborated code. When the teacher explains a

concept in elaborated code, it is as though a foreign language is being spoken to the lower class students.

In fact, Bernstein often mentions his debt to Benjamin Whorf, and his theories resemble those of Whorf in many ways. Lower and upper class language can be viewed as two different languages, and once a lower class child has learned a first language, his thought processes are fixed to such an extent that he/she cannot learn to think in the elaborated code language of the upper class.

We have seen in the discussion of the Whorf hypothesis that the hypothesis came under attack during the sixties, when weaker and weaker versions of it were proposed. Similarly, Bernstein's ideas were criticized for many of the same reasons. Bernstein subsequently modified most of his theories, although he is still known to most linguists for his early theory of elaborated code versus restricted code.

In his later work, Bernstein (1971-75) abandoned some of his earlier terms in favour of the notions of *position-oriented* and *person-oriented* families. In position-oriented families, children are expected to fall into line with the values and assumptions of the family (and thereby, class) to a much greater extent than in the person-oriented family, where children are expected to display a high degree of individual autonomy. In the conversation of a closely-knit position-oriented family, the thoughts of its members are more closely in tune, so that details need not be made explicit in the language. Context is strong, and therefore articulation is weak. This type of language is very similar to Bernstein's restricted code of earlier years. On the other hand, an independent child of a person-oriented family will be somewhat distanced from the context of the family, and will naturally need to make certain details explicit in language. Such language will therefore naturally tend to resemble the elaborated code.

For example, position-oriented language tends to use pronouns whose antecedents are not clear to the outside listener, since a family member who is on the same 'wavelength' as the speaker will be expected to pick out the person or object to which the pronoum refers. A person-oriented child, on the other hand, will tend to use the noun or person's name in order to make explicit to whom he is referring.

A second consequence of person-oriented language is that sentences must be planned and ideas linked in order to lead the listener through the message. This leads to the use of prepositions and other linking words. In position-oriented speech, ideas are

"strung together like beads on a frame rather than following a planned sequence" (Bernstein, 1971-5, vol. 1, p. 158.)

The implications for the classroom are the same as those of Bernstein's earlier theory: the position-oriented students will not be used to the explicit language of school. This would be especially true of the mathematics classroom, where the precision of language is particularly important.

In fact, most of Bernstein's early theory remains unchanged, in that he still claims that social conditions in the home provide the environment for a type of language which affects performance in school. In the later work, however, position-oriented families may or may not be from the lower class, although they tend to be. Thus, social class becomes a secondary rather than a primary determinant of school behaviour. A second difference is that the later work no longer claims that social background absolutely precludes the use of the elaborated code; children who have been brought up using restricted code will simply *tend* to use it more often. This change reflects the softening of the Whorfian hypothesis to weaker versions, in which the first language is not an absolute barrier to second language thought, but merely exerts an influence on it.

The difficulty with Bernstein's theories is that they are very difficult to verify experimentally. Even if it could be documented that children who use restricted code or who come from position-oriented families tend to perform badly in school, it would by no means prove that their language actually *causes* their poor performance. Let us examine some of the experimental evidence.

There is no shortage of evidence that lower class mothers use a very different type of language with their children than do middle class mothers. Middle class mothers appear far better at instructing their children how to do things. They encourage verbalization, use language to explain cause-effect relationships, and reason from analogies. Most of these studies, in order to be as quantitative and hence as 'objective' as possible, tend to count the number of times mothers or children use such words as prepositions or pronouns. A well-known early study by Hawkins (1969), for example, reported the stories of over 300 children from either working-class or middle-class families. He found the use of pronouns much more common among working-class children. He cites what is perhaps an overtypical example of the restricted code as follows:

> They're playing football and he kicks it and it goes through there - it breaks the window and they're looking at it and he comes out and shouts at them because they've broken it - and then she looks out and she tells them off.

It is easy to notice that there are almost no antecedents for the pronouns; the outside listener who is not aware of the context does not know exactly who 'they' are, or who 'he' or 'she' is.

In a similar experiment, Turner and Pickvance (1970) counted the number of question words such as who, what, why, and other expressions of uncertainty in the speech of sixteen-year olds. Middle and upper class children showed a higher frequency of such expressions than did lower-class children. This fact was interpreted as signifying a "greater awareness of uncertainty in certain areas of experience", and as "encouraging the children to be flexible in their thinking."

Dozens of studies of this nature could be cited, all of which report some type of difference in speech among social classes. They all share a logical leap in their conclusions: the claim that speech habits affect performance in school or cognitive skills. The counting of pronouns may quantify differences between classes of people, but it says nothing about their thought processes, nor does it shed light on the type of language the speaker is actually *able* to use. It is the old *post hoc ergo propter hoc* fallacy: lower class students use a certain kind of language - lower class students perform worse in school - therefore thier language causes them to perform badly in school. Labov (1972), perhaps the dean of modern sociolinguistics, has criticized this kind of experiment on this account: "the cognitive style of the speaker has no fixed relation to the number of unusual adjectives or conjunctions that he uses."

There are other dangers in the testing of Bernstein's theories. Several have been discussed by Trudgill (1975), who points out that one serious danger is to interpret Bernstein as claiming that lower-class language is actually *inferior* to middle class language. On this point virtually all linguists are in complete agreement: lower-class language displays all the complexity and nuance of middle class language, although perhaps in a different way. Lower-class language has its own codes and hidden meanings which are simply different from those of middle-class language. Teachers should not consider lower-class children as linguistically deprived; rather, the teachers should realize that

lower-class children bring a richness of language to the classroom, but simply one which is different from that of the middle class.

Another problem is that even if a child does not manifest a certain type of speech, that is no indication that the same child *cannot* use that type of speech. A child who tells a story to an experimenter in obviously restricted code may well be perfectly capable of using an elaborated code in other circumstances. Besides, even if a child is deficient in vocabulary, that vocabulary can be learned in school or elsewhere. Use of restricted code is no proof that elaborated code cannot be learned.

A study by Heider, et al. (1968) showed that with various amounts of adult prompting, lower-class and middle-class boys could describe a picture of a horse using roughly the same language style, especially in citing roughly the same number of attributes of the horse. A similar study by Palmer (1968) showed that if the experimenter had built up a rapport with the subject before testing, many linguistic differences between lower-class and middle-class students diminished greatly. Thus, lower-class children are not permanently locked into any linguistic style.

Trudgill sums up the results in very negative fashion:

> All the theorizing of the past sixteen years appears to have reduced to evidence that, in situations more artificial and alien to them than to middle-class children, working-class children use a higher proportion of pronouns. Is this what it has been all about?

Classroom discourse analysis

Classroom discourse analysis attempts to observe the underlying structure of a classroom situation. It is concerned less with nouns and verbs than with functions of sentences: soliciting responses, responding, guiding the direction of conversation, clarying, etc. Thus, it would be typical for a discourse analyst to observe that in the working-class home, a certain percentage of parent-child conversation is devoted to the mother's telling the child to do something, while in the classroom, a relatively higher percentage of the time is devoted to the child's asking questions, for example. Such differences would point to the fact that working-class children find themselves in rather alien situations in the classroom.

Most educators will be familiar with the work of Flanders (1970), whose methods of analyzing teacher talk was an early form of discourse analysis. His sorting of talk into categories such as who was doing the talking, whether sentences were questions or statements of fact, etc. has served as a model for much of the educational discourse analysis done since then. The methodology has been discussed at length, the major criticism being that the *a priori* choice of categories can easily bias the research. Researchers may choose categories in advance which will support their prejudices.

Without dwelling on the methodology of discourse analysis, we may still discuss what the analysts have learned about classroom language. Sinclair and Coulthard (1974) have investigated ways in which teachers structure their lessons by their speech. Teachers usually begin by establishing boundaries for what is to be studied. What follows is almost a game between students and teacher in which boundaries are tested in what the authors term *boundary exchanges*. Certain linguistic *moves* occur in this chess game, such as a *frame*, used by the teacher to suggest that the structure is or is not going according to plan, as evidenced by words such as O.K., well, now, so, etc. To plan for future moves, the teacher also uses a *focus*, which delineates the direction in which the lesson is headed. The students use those cues to decipher what the teacher considers important and what questions are likely to appear on the test. The students who best interpret the teacher's language are likely to be the ones who succeed in their schoolwork. Most students know that figuring out what the teacher is likely to ask is often even more important than learning the material in the lesson.

Despite the various scholarly approaches to the study of discourse of classroom activity, in my opinion the best description of what actually goes on is the down-to-earth approach of John Holt (1964). Holt realizes the strategies going on in the game between students and teacher. His major contribution is in pointing out that much of school is 'answer-centered' rather than 'problem-centered'. The most important goal of students is to satisfy the teacher with the right answer, regardless of whether any understanding has taken place, and many strategies revolve around getting the teacher to give away the answer.

> They see a problem as a kind of announcement that, far off in some mysterious Answerland, there is an answer, which they are supposed to go out and find. Some

> children begin right away to try to pry this answer out of the mind of the teacher. (p. 88).

Why are children so prone to play the answer game? Holt's answer is that they fear failure, both because of the disapproval of the teacher and because of the ridicule of fellow classmates. In this, lower class students may be at a disadvantage, since they are particularly sensitive to being told they are inferior. Thus classroom activity becomes an exercise in conflict avoidance:

> Game theorists have a name for the strategy which maximizes your chances of winning and minimizes your losses if you should lose. They call it "minimax". Kids are expert at finding such strategies. They can always find ways to hedge, to cover their bets. (p. 13)

Most of these strategies involve highly subtle observations of linguistic behaviour. Holt, among his rich store of examples, cites a maths teacher who, in attempting to "throw them a little curve ball.... keeps them on their toes", unconsciously tipped off the students by the expression of his voice that "something was coming up and that attention had better be paid." It is these subtle changes in tone-of-voice which are very difficult to measure analytically by, say, counting verbs, and which may be the keys to understanding classroom dialogue.

If the lesson does indeed resemble a linguistic game, its rules are not written explicitly. Those students who have best learned the unwritten rules will be the winners. Many of those rules will already have been learned in the home, especially if parents' language resembles that of the teacher. It is possible that many of these rules are specific to the middle class, and that lower-class children are at a disadvantage in school because they have not learned those same unwritten rules at home. A lower-class child may not perceive a verbal cue from the teacher that a certain topic is considered essential for the test, simply because the child may not be familiar with that particular rule of the game.

It is perhaps those subtle changes of tone-of-voice which are most indicative of class language. People of one social class may express nuances in tone-of-voice in different ways from those of another class. If the teacher is of the middle class, then middle class students will be best able to pick up the subtleties of the teacher's language.

It must also be noted that the mismatch argument is not claiming that lower-class students cannot *learn* the game rules of the classroom situation. By secondary school they may become quite proficient at reading the teacher's verbal cues. But given the fact that students are branded at an early age as to their likely success or failure, it is possible that failure to play the game by the rules in the first few years may well stigmatize children for years to come.

That these disadvantages exist is the subject of considerable speculation, and they have not been demonstrated conclusively. The theory of negative effects of a mismatch between the language of home and school remains just that: a theory. Some experts believe, in fact, that all the essential elements of the classroom are equally present in the homes of lower-class children. Maclure and French (1981) argue:

> Children most certainly do *not* enter a culture of unfamiliar interactional routines when they enter school. They have already encountered and participated in the majority of sequence types they are likely to be expected to share in school. This holds for a wide range of interactional activities. In fact, as far as conversational *structure* is concerned, the school setting may, if anything, present children with a *more limited* set of conversational options than they have become familiar with at home. (p. 227)

Thus even the theory of differences between home language and school language may be questioned. Even if it could be demonstrated conclusively, it would remain to be proved that such differences affected performance in school.

The language of mathematics

If mathematical language is in some ways a language of its own, one may ask whether it is more accessible to members of one social class than to another. Especially if we view mathematical language as a register, i.e. a specialized jargon, it is quite possible that this jargon is more readily accessible to one social class than to another. It has often been pointed out (e.g. Pimm, 1987) that much confusion in the mathematics classroom is due to confusion between the register of mathematics and everyday speech. A word

may have one mathematical meaning and a very different everyday meaning, and the students are not quite sure which one is being used at any given time. If, for instance, a teacher draws eyes, a nose, and a mouth on a cube and refers to the 'face of the cube', which register is intended?

Perhaps middle class children, having been exposed to more varied experiences, and thereby to more change in register, are more adept at discerning which register is being used. The rapid shifts between mathematical register and everyday speech may be more confusing to lower class children who have not had the experience of shifting registers in the home. This is all in the realm of possibility, and so it is necessary to pinpoint just what it is about the mathematical register which might render it more accessible to middle class children.

It might be possible to argue that mathematical language attempts to state its message in as few words as possible, and therefore that it resembles the restricted code of the lower-class in its short sentences, economy of vocabulary, and paucity of superfluous description. If we reconsider the little girl's description of the horse (restricted code) and the treehouse (elaborated code) cited earlier, we would surely say that the language of mathematics textbooks resembles the former far more than the latter. If so, we might expect lower-class children to perform relatively well at mathematics. The evidence, of course, shows that quite the opposite is the case: lower-class students tend to be particularly weak in mathematics. Why?

Mathematical language is very different from lower-class speech in at least one fundamental respect: it is decontextualized, in the sense discussed in Chapter 1. The mathematical statement stands on its own and does not need to be considered in any particular context. Lower-class language is highly contextualized, as evidenced by the use of pronouns and assumptions about what the listener knows. A mathematical statement should be understandable to people everywhere, while lower-class language is usually meant for the ears of someone who knows the speaker's situation already. A related feature is that lower-class language is highly personal and egocentric. It does not often portray situations as seen from different points of view. That is one of the main features of mathematics: the ability to abstract a thought and to place it outside the individual. The analytical mode of thought is to focus on one aspect of a question at a time in isolation from all other background noise. This mode of thought is thought by many to be alien to most lower-class children, who view all events only

in the context of the entire situation and how they relate personally to that situation.

This view does not necessarily substantiate the claims of Whorf and of Bernstein that it is the contextualized language itself which determines the thought patterns of the children. Rather, it could still be the case that the use of contextualized language is only symptomatic of a way of thought common to lower-class children. Thus, it would be a mistake to claim that children's mathamatical abilities could be improved simply by using more decontextualized language.

Still, it is possible, and I believe probable, that middle-class and upper-class parents who year anfter year use abstract terms, refer to cause and effects, and in general assume less about the context of situations, create an environment which facilitates the use of abstract mathematical thought by the child.

Summary - language deprivation?

The treatment of lower-class children in primary schools is often based on the assumption that they are linguistically deprived. Bernstein's theories are often cited to support this view. With this assumption in mind, programmes are developed to bring the level of the children's English up to standard. This is considered particularly important because the linguistic deprivation is thought to be at the root of deficiencies in many other cognitive skills, including mathematical ones.

The sociolinguistic research evidence gathered to date does not support the view of linguistic deprivation. For one thing, the notion that lower-class people speak an inferior brand of English has been discredited. Lower-class children do learn language skills at home, and can express themselves competently in situations in which they feel comfortable, using types of linguistic sophistication which do not *seem* sophisticated to middle-class speakers simply because they are different.

Secondly, the multitude of studies which have demonstrated differences in speech among social classes have yet to demonstrate that those differences are indeed the causes of difficulties in school. These studies never seem to pose the question, "Why are there linguistic differences?" If they did, they might discover some other factors which lead to both the linguistic differences as well as to school problems. In this way, the linguistic differences

would be seen only as secondary effects and not as primary causes of deficiencies at school.

I do not wish to propose a formal theory of my own; rather, I wish to describe the *type* of theory which might describe the situation. Suppose, then, that we were to focus on the emphasis in middle-class homes on individual advancement, as opposed to the lower-class emphasis on group solidarity. On one hand, it might be argued that group solidarity led to more context-oriented language, in which statements could be understood only by members of the 'in' group. The middle-class language would be more highly individualistic, and would resemble the elaborated code. On the other hand, school emphasizes individual achievement. Children are competing with each other for top marks, and collaboration on tests or assignments is considerd strictly as cheating. The less-goal-oriented lower-class children will apply peer pressure on their classmates not to excel, as individual excellence is seen as desertion from the group.

Labov's (1969) well-known conversations of black children from the New York ghetto seems to point to a theoretical base of the type just outlined. He showed that the proficiency with the complicated slang of Black English increased as the social standing of the child among his peer group increased. Conversely, children whose status in the gang was low tended to do better in school, possibly because the pressure to conform by doing poorly in school had diminished.

Whether or not this particular theory has any substance, it rests on a view that there may be fundamental factors in lower-class homes which lead simultaneously to a certain use of context-oriented language, as well as to poor performance in school, but that there may no direct, causal link between language use and school performance.

To be sure, schools preach the middle-class values of individual success and goal-attainment. Lower-class children may not readily buy into those values, but if they do, they must buy middle-class language use along with it. Whether they like it or not, they will be judged by future employers and educators according to their proficiency with middle-class language.

Teachers must keep these hard facts of life in mind. They should not consider lower-class students as linguistically deficient or inferior. Rather, they should regard linguistic diversity in the classroom as an asset, but should bear in mind that the subtle signals they send to their students may be picked up better by middle-class children who use those same subtleties at home.

Lower-class children may be speaking with subtleties of their own, but the teacher is not picking them up, either. Teachers should therefore not assume that lower-class speech is not subtle simply because they do not notice any subtleties.

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CHAPTER 11

ETHNOMATHEMATICS

Ethnomathematics is a term which has been coined in the 1980s to reflect the influence of cultural factors on mathematics and mathematics education. It therefore covers a very wide territory - indeed, far too wide for this single chapter to treat adequately. For a fuller discussion, the reader is advised to consult the excellent work of D'Ambrosio (1985), one of the prime movers of ethnomathematical research. Strictly speaking, the cultural aspect of mathematics education exceeds the borders of our discussions about linguistic factors only, but since language and culture cannot be separated, it is only right that a brief discussion of ethnomathematics be included in this volume. The treatment of the topic will of necessity be limited to ideas which relate to material already mentioned in this book, so many major topics must be omitted or only briefly mentioned.

The term 'ethnomathematics' is difficult to define precisely, and there is some disagreement as to its actual meaning. It is perhaps useful to begin with what ethnomathematics is *not*. It is not a collection of interesting folk games, measuring techniques, or counting systems used by various 'primitive' cultures around the world, although these topics may be of some relevance to certain ethnomathematical research. Nor is it a racist doctrine which states that different races have differing mathematical abilities. Rather, the term refers to a stance, perhaps even an ideology, that mathematics education should be relevant to the goals and aspirations of students, wherever they may be. It is therefore an examination, above all, of *values* - the value of what is taught and the value of how it is taught. It is a realization that mathematics, as it is taught in most of the world, is Western, middle- and upper-class mathematics and is permeated with certain Western values which may not be in harmony with the values of all societies. Educators have only in the past decade or so

fully realized that mathematics is not 'culture-free'. This notion has been dealt with rather thoroughly in the literature, and excellent discussion can be found in Bishop (1988) or Gerdes (1988a).

Most of the discussion in this chapter on culture will deal with developing countries and traditional societies, but many of the arguments can be applied to social classes and minoritiy cultures which have been excluded from the mainstream cultures within industrialized countries. At the heart of the matter is a questioning of why educational planners in developing countries must adopt the same curriculum and methods as those of the industrialized countries. One of the major questions is that of the language of instruction: should developing countries use their own local languages for mathematics education?

Values? Whose values?

If one is to tackle the question of the value of mathematics education, it is necessary to admit that there will be differing viewpoints on the subject. Educators may feel that mathematics education should develop the minds of students, while governments may feel that training for development is the value of education, and last but not least, the students themselves, along with their families may view education primarily as a step up the social ladder. The objectives supported by these three groups may at times be found to be in conflict.

The point of view of mathematics educators has been well presented in the UNESCO document on the goals of mathematics education (Morris, 1981). A complete discussion is impossible in the short space provided here. Suffice it to say that objectives may be classified into 'formative' and 'substance' objectives (Niss, 1981) The former refers to

> ...any objective oriented towards influencing the student's general personal characteristics and capacities. These would include his attitudes, his general mental capacities, his abilities to perform certain general activities, etc. To these ends mathematics is merely a vehicle. The subject matter is not itself important.

On the other hand, by substance objectives,

> ...we mean any objective oriented towards providing a student with those competencies for which mathematics as such is indispensable. The term 'mathematics as such' is not confined in its meaning to specific content or operations. It includes anything involving mathematical substance in one form or another.

But who is to decide which competencies are necessary? The farmer in an African village may have a different set of necessary competencies from those of a student bent on studying engineering. A wide range of necessary competencies exist, and these needs cannot all be easily met by a financially strapped educational system. Any system of priorities will advantage one group while disadvantaging another group.

As a result, mathematics education in developing countries often attempts to be all things to all people. In constructing an official syllabus, each member of the team of curriculum planners has his or her pet topics which they insist must be included. The resulting syllabus is often far too large to be covered adequately in the limited time available, and so the students do not actually master any topic sufficiently. But teachers are not at liberty to slow down in order to teach a few topics well at the expense of other topics. The upcoming standardized examination will cover all topics and the teacher will be criticized by the students, their parents, the administration, and the government if all topics are not covered.

Thus the syllabus emphasizes a multitude of substance objectives at the expense of formative objectives. Students must acquire a wide range of competencies at perhaps a superficial level, and do not have the opportunity to concentrate on general mental development. This is perhaps one reason for the abysmal failure of the 'modern mathematics' movement in much of the Third World: modern mathematics focused on formative objectives, but the material was treated as though substance objectives were at issue.

For example, the modern mathematics curriculum included arithmetic in various number bases. This was intended to enable students to understand the base ten system in greater depth: a formative objective. In practice, however, base five arithmetic, for example, was treated as a substance competency in its own right, as though addition in base five was in itself an important

skill. Students, teachers, and in fact everyone soon perceived the irrelevance of base five arithmetic as a substance objective.

To return to Niss's classification of objectives, it is also useful to distinguish between social objectives and individual objectives. There are 'needs of individuals in mastering their own lives' and 'needs of society as a whole'. These will be discussed in greater detail below, but in the context of the present discussion, the question of the language of instruction can be discussed with reference to these two types of goals.

How does the language of instruction affect the meeting of individuals' needs? This question immediately raises others, for instance, "Do students develop their general mental processes better in their own language?" The answer to this question has usually been a resounding and unquestioning "yes". One of the key documents in this area is the report of a 1951 UNESCO conference on vernacular languages in education, which concluded with "the proposition that education is best carried on through the mother tongue of the pupil."

Twenty years later, however, the UNESCO manifesto was no longer considered axiomatic. Another UNESCO conference in 1977 on language teaching now concluded:

> Organizing language teaching in plurilingual contexts is a complex undertaking, and its success is conditioned by a variety of linguistic, social, education, economic, geographical, religious, psychological, historical, political, ideological, and administrative factors.

Indeed, as pointed out in Chapter 8, research on bilingual education has shown that second language learning is not *necessarily* inferior to mother tongue learning, and that in some instances (e.g., some immersion programmes), bilingual education may give students certain cognitive advantages over monolinguals. So at least with respect to the formative objectives of the individual learner, the picture is far from clear, and the UNESCO conference was correct to conclude, "A model that functions satisfactorily in one context may not [necessarily] be transferred to another country."

With regard to substance objectives, the question of transfer must be raised: do skills learned in one language transfer to situations where a different language is used? If a student learns, say, multiplication in English at school, can he/she then apply multiplication to compute prices at the local market using the vernacular? As pointed out earlier, if the student has attained

a certain threshold level of competence in *both* languages, there should be no problem of transfer of skills between languages. But in developing countries, students often have not attained the threshold level in even one language, let alone two. There is some evidence that in such a case a second language skill may not completely transfer back to first language situations. This is one area where more research is necessary. A related question which has rarely been asked is, "How well do skills learned in a first language transfer to second language situations?"

If the objectives for individuals raise a set of complex questions, the objectives for societies raise even more difficult issues. A common struggle occurs between small groups who wish to preserve their ethnic and cultural identity through vernacular education as opposed to majorities who wish to unify a country through nationalism and education in the national language. In many newly independent countries, this politically sensitive issue was avoided by adopting the language of the formal colonial power as a unifying factor without imposing one local language on speakers of another. From time to time, however, some countries have seen the colonial language as a continuation of Western dominance, or 'neo-colonialism', and have opted for a local language: Kiswahili in East Africa, Bahasa Indonesia in Indonesia and Bahasa Malaysia in Malaysia, etc.

Although the adoption of local language instruction is usually seen as a liberation from the colonial power, it should be pointed out that such is not always the case. It can work as an isolating factor which denies students the right to broader access to the international community. For example, in South Africa, the Bantu Education Act of 1953 forced the black population to learn in their own languages. This had the effect of entrenching apartheid by isolating and fragmenting the various black ethnic groups. If Sothos and Zulus, for example, were not able to communicate either with each other or with the outside world in some common language, there could be no unified opposition and the anti-apartheid case could not be presented to the outside world.

On the other hand, in many countries ethnic groups actively *seek* isolation from majority groups. They fear that education in the language of the majority will lead to the assimilation or destruction of their culture. They may even reject instruction in a foreign language such as English because if their children learn English, they may forsake their traditional life, lose their traditional values, and migrate to the city.

This complex issue of the choice of language is made even more intractable because it can rarely be discussed rationally. Nationalism and ethnic identity raise strong emotions which often obscure important facts and logic. The results of academic research and the voices of educators are unlikely to be heeded by policy makers.

Government objectives - the hidden curriculum

Since the government is likely to be the dominant maker of educational policy, it is necessary to consider questions of mathematics education policy from the governmental point of view. As pointed out above, the issue of national unity has been a high priority in developing countries, and so the predominant pattern has been to adapt a single language of instruction and a common mathematics curriculum across the entire country. At independence, most of the new rulers had been educated in the former colonial power, and so they were familiar only with the educational system of those countries. It is not surprising that the newly independent nations adopted Western styles of education.

What *is* perhaps surprising is that thirty years after independence, the educational systems in many developing countries have not changed very much. Governments have had sufficient time to redesign their curricula and to implement education in local languages, but on the whole they have chosen not to do so. Why?

We can point to many reasons for the lack of change. Certainly the financial constraints on producing new textbooks and retraining teachers are enormous. But if education were a priority, funds could be diverted from, say, military expenditure, to these purposes. It may be the case, rather, that governments do not wish to change. This is what many supporters of the ethnomathematics movement claim. Their claims have their roots in the more radical ideas of the 1960s, notably those of Ivan Illich (1971).

In Illich's view, the educational system is the backbone of an elitist society. Through it, the upper class may perpetuate its dominance. Education in a foreign language is one of the best methods of retaining the power of the elite. If the lower classes are unable to speak or to read and write the language of government, they are effectively excluded from the political life of the country. In some countries, the leaders at independence, trained in Europe, rose to positions of power precisely because of

their education in the European language, and so they were not eager to disturb the status quo by changing the educational system in their own country.

In other countries, the colonial language was abandoned in favour of the language of the locally dominant ethnic group, but the effect was the same: the ethnic group used language policy in education to maintain its dominance over other groups. For example, in Malaysia, the dominant Malay element was able to make Bahasa Malaysia the medium of instruction. Malays thereby obtained an advantage in educational achievement over the Chinese population, who were forced to learn in a second language very different from their own.

But even more benevolent governments have seen that if their countries are to become part of an international community, they must promote an international language such as English as well as an international curriculum. This is particularly true in mathematics and technology. Governments have almost uniformly accepted the doctrine that they must no longer rely on production of a single commodity, but that they must diversify and increase productivity through advanced technology. One of the greatest needs in developing countries is that of a population well-trained in science and technology.

Developing countries have therefore placed a heavy emphasis on mathematics education as necessary for the development of technology. In the drive to keep up with the West in technology, it is necessary to teach a Western-style curriculum in science and mathematics. And since the books on these subjects are written in Western languages, there is additional pressure to adopt the foreign language as the medium of instruction.

The industrialized nations are of course not eager to change this situation. They are the greatest advocates of technology as the panacea for development. Indeed, can they not point to the rise of Japan as directly due to Japan's adoption of Western technology? The developed countries of course have a vested interest in maintaining a situation in which they hold the high cards.

Moerover, the West has been able to maintain this dominant position through 'aid' to education. Expatriate teachers are sent to many developing countries, together with Western textbooks, advisers, journals, and educational technology. It is usually the Western advisers who design the curricula and educational policy of developing nations. These experts cannot be entirely blamed, for they are not consciously fostering domination. They resolutely affirm what they have been taught in their own countries, and have

no choice but subconsciously to reflect their countries' educational policies. If the developing country has no locally trained technologists, they must rely on these foreign experts' advice or on nothing at all.

But it is often through such a process that the curriculum contains what Ivan Illich has termed the 'hidden curriculum'. Certain ideals are taught implicitly; for example, the importance of science and technology is fostered by its emphasis in the curriculum. Illich lists other hidden items:

> Everywhere the hidden curriculum of schooling initiates the citizen to the myth that bureaucracies guided by scientific knowledge are efficient and benevolent. Everywhere the same curriculum instills in pupils the myth that increased production will provide a better life. And everywhere it develops the habit of self-defeating consumption of services and alienating production, the tolerance for institutional dependence, and the recognition of institutional rankings. (p. 106)

In science and mathematics education, the list must include a respect for science, mathematics, and technology, along with the doctrine that problems should be solved and decisions made based on quantifiable data. In Europe, science has been questioned, but many developing countries still cling to the worship of technology. The case against science has been made in no uncertain terms by Feyerabend (1975):

> Scientific 'facts' are taught at a very early age and in the very same manner in which religious 'facts' were taught only a century ago. There is no attempt to waken the critical abilities of the pupil so that he may be able to see things in perspective. Science is excepted from criticism. In society at large the judgement of the scientist is received with the same reverence as the judgement of bishops and cardinals was accepted not too long ago. (p. 157)

If science is indeed a religion, what, then, are its basic tenets? First, there is the dogma that the scientific method is *the* correct way of finding truth and of achieving results. In this method, observations are made which lead to a scientific 'theory' which explains the general pattern of the observations and can be

applied to new situations. Related to this method is of course a fundamental belief in a universal logic (Bishop, 1985) as more important than traditions, experience, or personal status.

The scientific method leads to the formation of theories. However, scientific and mathematical theories are not truth in themselves, but simply models constructed to deal with certain problems. In equating theory with truth, the theory is no longer open to modification. For example, Newton's theory of mechanics and calculus remained for a long time a useful model for describing how bodies move. But if it had been allowed to become 'truth', the theory of general relativity might never have experienced the modification into Einstein's special relativity. The old model was adequate for describing the motion of large, slow bodies, but needed to be altered to describe sub-atomic particles moving at nearly the speed of light.

The distinction between truth and usefulness has led to the notion that the usefulness of science is only relative to specific situations. If science were 'true' in an absloute sense, then it would be universally true and thereby universally applicable. Developing countries have turned to 'appropriate technology', which demystifies Western results as not universally true. But it retains the worship of the scientific method and the belief in technology as being the best, if not the only, useful method of development. Mathematics has so far been spared, and is still regarded as 'true'. But the ethnomathematics movement is leading to an 'appropriate mathematics' movement which will value certain mathematical tools for their usefulness rather than for their 'truth'.

The relationship between mathematics education and religious truth has recently been explored in detail by Jurdak (1988), who describes, in particular, incompatibilities between the mathematico-technological (MT) culture and the Islamic-Arab (IA) culture. In the IA culture, scientific logic takes a back seat to tradition, especially in the form of quotations from the Koran, whose truth may not be questioned and which may not be subjected to the scrutiny of scientific experimentation. Moreover, the method of teaching in Koranic schools emphasizes rote memorization of the Koran and other texts whose authority must be accepted on faith. In the MT culture, the emphasis is on student participation in finding rules and patterns, as well as on questioning assumptions or analyzing logic.

Jurdak goes on to point out some of the related cultural implications of using a foreign language as the medium of instruction:

> It seems to me that, at the basis of this, is an implicit assumption that the language of mathematics is culture-fair because it is culture-free. As a consequence, the language of instruction, whether native or foreign, does not have cultural implications. This assumption may be challenged in the same way one challenges the universality of mathematical concepts. As soon as mathematics is applied in problems and situations, the language of instruction and learning becomes a cultural carrier in terms of behaviors, social relations, habits, and values. (p. 5)

For example, the Arabic language is considered the carrier of the IA culture, and so, when a foreign language like English is used, it prevents the transmission of the IA culture. The issue of the language of instruction becomes an issue of IA versus MT cultures. Certainly the Islamic world will continue to want Western technology, but there is considerable debate as to how far that technology will erode traditional Islamic values. Economic questions become confused with ideological questions.

At some point, probably later rather than sooner, governments will realize that in playing the economic GNP game they are competing with the industrialized countries for markets and that they are fighting a losing battle. The economic superpowers have too great a head start in productivity, technology, and education. Almost anything the Third World can manufacture, the West can manufacture more cheaply and efficiently. The same argument applies to education, if education is to be viewed as a product. With vast sums of money to be spent on science laboratories and libraries, the West will surely outstrip the developing countries, particularly since students in many countries must learn in a foreign language and must adapt to a cultural bias in the material they learn. They simply cannot compete.

My experience in several developing countries is that schools and especially universities often attempt to follow mathematics syllabuses similar to those in Western universities and to 'keep up standards' so that their graduating students can be claimed to be on a par with Western gradutates. Since this is

generally impossible, the goal of the teacher is to often water down the material to a level where the students can claim that they have studied such and such a topic, and can throw around a bit of the mathematical vocabulary without understanding the material at all.

The alternative is for governments with scarce resources to realize that they cannot offer the same standard of education as exists in developed countries. What they can do, however, is to identify a few important skills which they want their students truly to master rather than attempting to cover vast syllabuses superficially.

What do students want?

For the vast majority of children in poor countries (as well as poor sections of rich countries), education means one thing: an opportunity to climb out of poverty. They probably care little about which subjects are taught or even how they are taught. The curriculum is dictated from above and is accepted as a hurdle which must be surmounted for upward social mobility.

Because education is so closely connected with socioeconomic stratification, schools sometimes become one of the principal instruments of an elitist system. It is obvious that not everyone can rise to the top of the social pyramid, and so the 'mathematics for all' movement would probably not even be accepted by the masses. If there were no competition for places in schools, it would take away from poor people one of the only means they possess for bettering their lot in life.

In a world seen as unfair by millions of poor people, the educational system is usually seen as one of the fairest institutions in that all children have an equal opportunity to succeed. If they work hard, children are told, they can someday expect a good government job and a Mercedes. They feel that they can work just as hard as rich children and are competing on an equal footing with all children.

In reality the competition is not equal, for the rich children have all the benefits of living in a culture more attuned to that of the educational system. Their parents value education highly and will spend money for private tuition; they may even speak the language of the instruction, for example English, in the home. For whatever the reasons, the hard facts are that rich children succeed in school to a far greater extent than poor children.

But this perception of equality remains, and has an effect on attitudes towards what goes on in schools. In order to maintain the dream of success, education retains a mystical aura. Mathematics in particular is a mysterious realm, and its mystery is increased if it is taught in a foreign language. The students would not want it any other way, for a de-mystified mathematics, brought down to the level of everyday rural tasks, might shatter the dream of escaping from that environment. Educators may complain that the familiar calculus problem of the melting snowball, found in most textbooks, is irrelevant to students in tropical countries, but the students love it, since it hints that one day they may have the wherewithal to study in Europe. By the same token, studying in a European language perpetuates that same dream.

As a result, it is easy for students to accept the policy of learning many mathematical things badly rather than a few things well. A major objective becomes the ability to use the mathematical jargon to impress and mystify both oneself and others, and the mystification is heightened if the jargon is in a foreign language. Schools and universities are turning out graduates who can talk about a wide range of mathematical topics in a foreign language, but who are incapable of applying it to their lives. The same might be said even for education in developed countries, and education in students' first language.

A second belief of the educational institution, especially in mathematics and science education, is that only quantifiable, measurable quantities are valuable. In fact, there is a tendency to believe that unquantifiable entities do not exist. Consider the lack of academic interest in the occult and ESP. The most common argument against the existence of ghosts, for example, is that ghosts do not readily submit to scientific experimentation.

In many cultures, the citing of numerical statistics is not considered a very convincing argument, simply because numbers do not carry much weight in those societies. These societies usually do not have highly complicated number systems and are therefore labelled 'primitive'. On the other hand, quantifiable data are the only acceptable results in an academic debate in a schooled society. We have only to remember how the United States Government used the weekly 'body count' in Vietnam - 4,265 Viet Cong, 13 Americans - to reassure the populace that the war was being won.

The emphasis on measurement has its drawbacks even in developed countries' schools. Teachers tend to emphasize skills which can be taught in a lesson or two and which can then be

assessed. An algorithm such as solving a quadratic equation can be taught relatively quickly, and when students demonstrate their mastery of the algorithm, their teachers feel proud of having successfully taught the students. Long-term competencies such as critical thinking cannot be easily measured and so are not stressed by teachers.

A third tenet of the new religion is that students do not know what is in their best interests and must be informed of what is good for them by professional educators. They blindly accept the notion that mathematics is good for them. It may be true that a degree in mathematics may earn its holder a good job, but there may be cultural groups who simply are not interested in mathematics or do not hold it in such high esteem as educators do. It is slightly encouraging to see that some intractable minority groups such as American Indians do not always accept that dogma. Attempts are continually being made to induce them to buy into the mathematical value system; the following example is taken from an abstract of a study done by Arizona State University (1982):

> A 2-year longitudinal/cross-sectional project investigated factors related to highly able Mexican Americans' decisions to avoid or take advanced mathematics. The document concludes with suggestions for intervention strategies for increasing mathematics participation of all highly able students.

Or similarly, a report by Ortiz-Franco (1981):

> A mathematics-science program for precollege minority students, specifically Chicanos and American Indians, is based on the experiences of intervention programs designed to increase the participation of non-minority underrepresented groups in mathematics related careers.

These two examples illustrate how mathematics educators assume they know best that mathematics is good for the minorities, if only the minorities would realize that fact.

Changing the system

How can this apparently self-perpetuating situation ever change? Certain myths will have to be dispelled, for example the myth that one can only learn something if it is taught by a professional teacher. If teacher shortages become too acute in some countries, students may undertake to teach each other. A student who is good in mathematics may trade his or her knowledge with a student who is good in English, and they may help each other in a sort of educational barter system. If such a system became widely acceptable, students who wanted to learn or teach some specialized topic could form networks, for example, in the classified advertisements in the newspaper. These networks are what Illich (op. cit., 1971) in his classic book, *Deschooling Society*, called 'learning webs'. These would go a long way towards fostering the ideas in students' minds that they are responsible for their own education, that teaching and learning are two sides of the same coin, and that an individual may choose what he or she wants to learn.

But learning webs would only be a first step, since they would still be playing the game inside the system of examinations and standards set up by the institution of schools. As long as students are convinced that the school system knows what is best for them, they will be at the mercy of bureaucratic manipulators of their lives. According to the Illich school, it will only be when students perceive that they have been duped that they will begin to decide what is best for themselves.

It will probably be a long while before students begin blaming the system instead of blaming themselves. Society is so permeated with the value of schooling that if students cannot do mathematics, or if they drop out of school, they immediately place responsibility for the failure on themselves. The change in attitude will probably come from those students who have played the school game successfully only to discover that they are unemployable. Already in many countries, bright, industrious young men and women are beginning to find that a university degree is not an automatic meal ticket. They often feel they have been cheated, but as often as not they are so imbued with the value of schooling, they feel only that the remedy can only be found in *more* schooling, for example a Master's Degree or higher diploma.

But as long as societies place such a high value on science and technology, students who can complete a course in mathematics will probably be rewarded with a place in the elite system, and

will not rock the boat. The only way for the technological mindset to be shaken is if it is seen to be failing. This is actually happening in many parts of the Third World. Africa in particular has invested heavily in education and technology and has bought into the value system of consumption, production, development, and rise in GNP. Now, as living standards are declining, a few Africans are realizing that they have been induced to compete in a game where the probability of winning is small.

An unbiased assessment of what schooling actually accomplishes is necessary. The operative word is of course 'unbiased', which implies that the assessment should not be done by the school system. Most evidence gathered so far has come from educators who build the hidden curriculum into their studies. There have been many Piagetian studies, for example, which attempt to measure the development of children in various cultures. The results of these studies often tacitly conclude that if children in one culture reach the formal stage of thought later than those in another, there is something deficient about them. Value is placed on formal thought by the educators, who conclude that educational changes should be carried out in order to speed up the pace of development.

There are, however, some interesting results which seem to indicate that schooling does have a powerful effect on the mental development of children. A well-known study is the work of Greenfield (1966) among the Wolof in Senegal, who showed that Western schooling is perhaps the most important influence on the development of children's thought. This study has been supported by others (e.g. Hendrikz, 1966, in Zimbabwe; Prince, 1968, in Papua New Guinea; and Pinard, *et al.*, 1969, in Rwanda). On the other hand, several studies have not found schooling to affect Piagetian development (e.g., Goodnow, 1962, Waddell, 1968). In Liberia, Cole, *et al.* (1968) found large differences between schoolchildren and unschooled children on some tasks only, for example recognition of geometrical shapes and came to the not unexpected conclusion that

> the overall superiority of the schoolchildren can be accounted for by those problems which have the most relevance to their schoolwork. We should expect the schoolchildren to excel on those problems where recognizable geometrical shapes form a basis for solution. Thus, the overall superiority of the

> schoolchildren must be based on problems which have received special attention in the schools. (p. 165)

The educational system, therefore, is able to declare what things it considers important, and then to succeed at instilling those virtues in children. No wonder, then, that schools are seen to succeed, since they set their own criteria for success.

Teaching methods

Illich (1971, *op. cit.*) notes that teaching methods are remarkably similar around the world.

> Schools are fundamentally alike in all countries, be they fascist, democratic, or socialist, big or small, rich or poor. This identity of the school system forces us to recognize the profound world-wide identity of myth, mode of production, and method of social control. (p. 74)

This amounts to an attitude that if educational institutions around the world have similar notions of what graduates as packaged products should be like, then the methods for producing that uniform output should also be uniform. Indeed, even many of the calls for changes in teaching methodology according to cultural differences still assume that those modified methods are but different means to the same end.

For example, a report on practices in a Hawaiian school (Cazden, *et al.*, 1980) cites the use of the particular discourse structure of Polynesian conversations as beneficial to the educational process. The discussions followed what Watson-Gegeo and Boggs (1977) called a 'talk-story', in which the story is told by a string of narrators whose narrations overlap, and who are often interrupted by audience comments. In discussion groups or reading groups in school, this type of talk-story structure was encouraged in order to make the children feel more at home in the classroom. After the change in style, "the children's reading achievement increased dramatically. Researchers attribute this dramatic growth in reading to the children's fluent use of the talk-story discourse structure as an aid to reading comprehension." Thus, the universal goal of reading comprehension laid down by the educational system was never questioned. Use of the talk-story

was simply a device used to channel the children into the goal-oriented behaviour. Fluency in use of the talk-story could not be considered as a goal in itself.

In the article just discussed, the authors describe other projects which seem more hopeful in terms of the goals of education. A bicultural project (Klienfield, 1979) in an Eskimo school in Alaska observed dedicated teachers and even volunteers who mixed their factual presentations with "personal experiences of the teacher, references to Eskimo village life, delightful in-jokes, and broad humor". The classroom and teachers were part of the children's daily cultural experience. What is most promising is the method of assessment, which was according to local values as well as according to the usual formal testing. The school was seen as successful not only because of the students' subsequent success in college, or because of their verbal ability, but also for their 'psychic well-being' and their ease at adjusting back to village life, often in leadership positions. Thus, the very *goals* of the children's education were considered, not just the *method* of attaining the usual pre-packaged goals of formal education.

This is an important distinction. Many educators view the ethnomathematics movement as an attempt to use local and culturally relevant examples in order to get across mathematical ideas, for example, the use of Tchokwe sand drawings described by Gerdes (1988b) to teach many arithmetical and geometric ideas such as arithmetic progressions and the Pythagorean Theorem. This is not enough, as Gerdes is well aware; before teaching such notions, teachers must ask themselves whether the Pythagorean Theorem is of *value* to the children in their own cultural context.

Another aspect of the Western style of education in most countries is that only a limited number of individuals will succeed. There is necessarily a competition for places in secondary schools and universities. But individual competition is not accepted as valuable in many cultures. Traditional cultures often value cooperation, group solidarity, and obedience to the clan. This is often seen as an obstacle to educational progress, since an individual child may not express the desire to excel. It may be socially disastrous for a child to rank first in the class.

An example is cited by Erickson and Mohatt (1981), who describe an Ontario Indian school in which an Anglo teacher starts his classes in his usual style of seating the children in rows and asking individual children questions. But after some time, he found it necessary to seat the children in groups at tables and to call on groups rather than individuals.

The linguistic discourse structure of the classroom may also be affected by cultural differences in 'power distance'. In high power distance cultures, such as in the Far East, the teacher is the all-powerful dispenser of knowledge and must not be questioned by students. Teachers who go to East Asia to teach for the first time are often dismayed by the lack of classroom response by the students. The teacher may ask questions and get no answer. Students will not ask questions, make comments, or enter into any class discussions whatsoever. In other low power distance situations, such as the Eskimo school described above, it is important for children to take up the narrative, and to comment freely on the teacher's statements.

Teachers must adapt their styles according to the cultural styles of their students. In an Asian school, for example, students will most likely be expecting a formal lecture, and even if the expatriate teacher is used to a more informal, discussional style, he or she should adapt to the less interactive practices of Asian classrooms.

There are a host of other cultural differences which may influence the method of teaching, too numerous to describe here. Many of these differences fall into the category of differences in 'cognitive style', a term used to describe a variety of culture-related ways of looking at the world. The notion that people from different societies had different styles of coping with the world drew considerable attention in the 1960s due to the work of Witkin (1967). Witkin showed, for example, that in some cultures people are more adept at being able "to perceive an item as discrete from its organized ground when the field is structured, and to impose structure on a field, and so perceive it as organized, when the field has little inherent organization (structuring)" Naturally, this perception of order and structure should have strong consequences for the learning of mathematics.

Extensive research was carried out on such 'field dependence' (using Witkin's Embedded Figures Test and other measures), and this was shown to be a consistent culture-specific feature of a person's mental make-up. Correlational studies showed that field-dependence was related to certain aspects of intelligence (e.g., the Block Design, Object Assembly, and Picture Completion sections of the Wechsler Intelligence Scale), but only slightly with verbal skills and such quantitative abilities as Digit Span, Arithmetic, and Coding.

Whether or not field dependence is a separate factor from general intelligence has been debated, since factor analytical

studies have obtained different results. Satterly (1979) and Witkin, *et al.* (1977) found that field dependence loaded onto a different factor from general intelligence, while Roberge and Flexer (1981) found no such distinct factor, but observed that scores on the Embedded Figures Test correlate with tests of mathematical computation, concepts, and problem solving at .45, .47, and .48, respectively. A second study by Roberge and Flexer (1983) showed that field-independent students generally scored significantly higher than field-dependent students on mathematics achievement tests in concept attainment and problem solving, but not on computation.

Field dependence is the most commonly discussed type of cognitive style, but many others can be listed (e.g., Jones, 1986): kinetic vs. tactile, visual vs. information processing, and person-oriented vs. object-oriented styles. Perhaps the best known research in the domain of organizational behaviour is the work of Hofstede (1984), who posited a four-dimensional theory of cognitive style, according to which cultures may be assessed on scales of power distance ("the extent to which a society accepts the fact that power in institutions and organizations is distributed unequally"), uncertainty avoidance ("the extent to which a society feels threatened by uncertain and ambiguous situations"), individualism (as opposed to collectivism), and masculinity (which includes "assertiveness, the acquisition of money and things, and not caring for others, the quality of life, or people.").

Although the effects of many of these differences on mathematics learning have not been investigated in great detail, Jones reports that such differences

> may place black students at odds with the dominant style expected in school curricula, especially in math and science. However, black students are more likely to excel in math/science and choose science-related careers if they establish a strong relationship and perhaps role identification with a math/science teacher.

Without discussing all these differences in cognitive style and their various effects on mathematics education, it is possible to sum up the research by stating that strong differences in cognitive styles have been found among different cultures, that measures of these styles correlate significantly with measures of mathematical competence, and therefore that teachers should adapt their teaching methods to the cognitive styles of students. In so

doing, however, care must be taken to consider some of the cultural goals of the students (e.g., group solidarity, relation to community) and not evaluate teaching methods solely on the basis of tests of mathematical achievement.

Illich in the eighties

Most of the ideas discussed earlier in this chapter were influenced by followers of Ivan Illich around 1970. This was a period of radical thought which questioned most, if not all, of the institutions of post-Vietnam Western society. Much of the furore has now died down, but it is interesting to read what Illich and his followers are saying now, nearly twenty years later.

In his most recent book (Illich and Sanders, 1988) he in no way repudiates what he said in the seventies, but he focuses on a new factor which he views as important to the mentality of modern man, namely the language factor. In particular he is interested in the effects not of language on science, but of science on language. He believes that our technology-obsessed culture is radically changing the very nature of language. He laments

> ...the impact that communication theory has had on our colleague's perceptions of reality, turning the English language into no more than a code. Literacy is threatened as much by modern education as by modern communication.

Since science and technology have acquired the aura of absolute truth, as Illich argued two decades ago, the language of science has been used in many non-scientific contexts to convey that same aura. Illich calls these terms 'amoeba-words', and the type of language which uses them, 'Uniquack.' As an example of an amoeba-word he cites 'energy', which has deviated from its original definition in physics, as well as from its pre-physics definition, to a vague notion of a general potential to perform work, be it by a labour 'force' or by an automobile, and we speak, for instance, of society's 'energy needs' without knowing precisely what they are.

> The word 'energy' in this context is used neither with common sense, nor with the senseless precision of science, but almost like a sublinguistic grunt - a

> nonsense word. Energy, like sexuality, transportation, education, communication, information, crisis, problem, solution, role, and dozens of other words, belong, in this sense, to the same class. (p. 106)

Paradoxically, then, mathematics and science, which aim at total precision of vocabulary, have led society to use their same terminology in imprecise and even meaningless ways.

What is worse, this new form of language, similar to Orwell's Newspeak in *1984*, is used to manipulate opinion in a dangerous way. Since the 1970s, Illich has come to see not just technology and schooling as the enslavers of society, but now the language which they have generated is seen to have an enslaving power of its own. Amoeba-words can have a strong emotional effect and appear to say profound things when in fact they have no meaning at all. Demagogues can use the word 'love' to mean 'hate', that is, they can take a word which has lost its meaning and to twist its emotional aura to convey any meaning they wish. Another way of stating it is to say that modern language has become all connotation and has lost all denotation.

To Illich, the reason for this loss of meaning relates to the major differences between language and mathematics which were discussed in Chapter 1. Mathematics and language both are structured systems, but language is much more than just a structured system. The effect of Chomsky, through his addiction to the technological-mathematical world-view, was to reduce language to simply a code, and man to simply a machine. Today people are 'programmed' (a typical amoeba-word) for their 'role' in society. There is something dehumanizing about referring to learning as 'information processing'.

Mathematics teachers may claim that they are not responsible for Uniquack, for they insist on precise language from their students. But mathematicians and especially computer scientists often use their own specialized register (= jargon) in everyday situations. For example, mathematicians use the expression 'less than epsilon' to describe anything small, and in this way epsilon becomes an amoeba-word. Computer scientists use the word 'system' to describe just about everything under the sun, to the point where the term has lost virtually all meaning.

Are teachers all that evil?

Illich and his followers have been quite negative about schooling and, as we have seen, their negativism filters down to harsh criticism of teachers as agents of that evil institution. Mathematics teachers, in supposing that they know what is best for students, instill the ideology of the omnipotence of technology and science.

But is it all that terrible to care about children and wish to help them? It may be that the alternative to claiming to know what is best for a child is to disregard the child completely, since any attempt to help a person presupposes that the helper has something to offer. Children do need guidance, and so teachers do have some responsibility for guiding their students, and that guidance implies that teachers know something of value to impart to their students.

The answer to this dilemma is that teachers must be able to teach mathematics without advertising it, i.e., without preaching a gospel that mathematics is good for the soul. It is only natural for members of any profession to believe that theirs is the most important profession, but mathematics teachers must realize that mathematics is not important in the lives of many children, especially children in non-industrialized cultures. It is permissible to point out to students that modern society values technology and that success in mathematics is likely to lead to a good job, but it is not correct to preach that mathematics and technology are good in some absolute sense, or that the technological, consumer-oriented lifestyle is the ideal one.

This is similar to the answer provided by Feyerabend (*op. cit.* 1975):

> 'Strengthening the minds of the young' means strengthening them *against* any easy acceptance of comprehensive views. What we need here is an education that makes people *contrary, counter-suggestive without* making them incapable of devoting themselves to the elaboration of any single view.
>
> By now everyone knows that you can earn a lot of money and respect and perhaps even a Nobel Prize by becoming a scientist, so, many will become scientists. They will *become* scientists *without having been taken in by the ideology of science*, they will *be* scientists *because they have made a free choice.* (p. 164)

The message of this chapter is therefore that although people all over the world have mathematical needs, these needs vary from culture to culture, and these needs should not be mistaken for a single *NEED* for mathematics and technology in general. Mathematics teachers do have something to offer, namely a knowledge of mathematics, but they should be able to offer it without preaching its value. Ethnomathematics represents the view that mathematics can be taught in consideration of societal goals and needs other than those commonly associated with Western industrialization, production, consumption, and technological 'progress'.

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CHAPTER 12

APPLICATIONS AND CONCLUSIONS

In this final chapter I wish to present some of my personal views which have arisen out of my teaching and research experience, and to apply some of the theoretical conclusions of the literature to my views. The second part of the chapter is then an attempt to tie up the various strands of the book into some general conclusions.

Mathematics for non-Westerners

When I first went to teach mathematics in Africa, students raised three issues with me:

> 1. "We African students do not speak a European language as our mother tongue. We cannot be expected to be able to learn mathematics in English."
>
> 2. "The mathematics we are taught in school is Western mathematics and as such is foreign to our culture. We cannot be expected to appreciate the products of colonial cultures."
>
> 3. "Even if we could understand Western mathematics, why should we be forced to do so, as it is not part of our cultural heritage?"

These three issues must be taken seriously by anyone who participates in international mathematics education, especially by anyone who teaches the usual Western mathematics to non-Western students using European languages. I have struggled with these issues while teaching in various non-Western cultures for fourteen years now, and many of the ideas expressed in this book

are manifestations of my experience. My conclusions therefore necessarily contain an element of subjectivity.

The first issue above is simply a statement of the strong Whorf hypothesis, which has been discussed earlier. I do not believe the strong version of the hypothesis, which would state that it is *impossible* to learn mathematics in a foreign language. Indeed, there are plenty of fine African mathematicians who have succeeded in learning mathematics in English or French. Many of my own students have done well in mathematics, so at least success is possible.

Whether or not the language hinders the learning process in a second language is another matter, but I have not seen solid evidence to support even the weak version of the Whorf hypothesis. My experiments have shown that students with a good command of English as well as their own language do just as well on mathematics tests as students who speak English only.

My experience supports the threshold hypothesis and the concomitant interdependence hypothesis: if a certain threshold level in both L1 and L2 is attained, the bilingual student can learn mathematics just as well and perhaps even better than monolinguals. The catch is that the threshold is not often attained, especially in developing countries where resources for education are scarce. Many of my students in such countries, even at university level, have a completely inadequate command of the second language, and even poor command of their first language. For them, of course it is difficult to study mathematics. If we take this logic to its extreme, we can see that if students do not know any English *at all*, they will naturally have difficulty learning mathematics in English. That effect has nothing to do with the Whorf hypothesis.

The interdependence hypothesis states that if students are deficient in either their L1 or their L2, that deficiency may cause negative interference effects, and these students may even perform *worse* than monolinguals. I believe I have observed this effect in many of my students, but I am not sure whether the effect stems from a deficiency in the L1 or in the L2, or both. Many of them were weak in English, it is true, but I am disturbed by my finding in Lesotho (Zepp, 1982) that by form 5 of secondary school, students whose English was only fair performed better on a logic test in English than they did in their L1, Sesotho. Many who had taken the test in Sesotho even stated afterwards that they had translated the questions into English first.

The inability to transfer mathematical principles learned in L2 to L1 situations suggests that perhaps something is deficient in their L1 abilities. When the same experiment was repeated with Chinese-speaking university students, there was no apparent difficulty in the transfer of logical principles from L2 to L1. I attribute the discrepancy between the Sesotho result and the Chinese result to the fact that Chinese students have a strong component of their education in Chinese, and that they have therefore attained the threshold level of competence in both English and Chinese.

It is once again useful to bring in the distinction between BICS (basic interpersonal communicative skills) and CALP (cognitive/academic language proficiency). If in fact these two proficiencies are different skills, as the research literature tends to support, then teachers may be making a mistake by assuming that if children communicate well in their own language, they should be able to study academic subjects at secondary or university level in their own language. My experience with Sesotho speakers shows that they have not developed CALP skills in their first language. Educational systems should be training children in these skills. This can be done not just by making students study their own language; instruction in other subjects should be done using the L1 as well.

The success of many immersion programmes leads me to believe that developing countries should attempt programmes along the same lines. It has been demonstrated that at least in some cases students can learn in two languages and can become proficient in both in the same amount of time as monolinguals, with no apparent damage to cognitive development.

One key to the success of immersion programmes has been the attitude that both languages are considered important and share equal prestige. This has not been the case in many of the developing countries in which I have taught. There, the colonial language is the language of power and prestige, while the local 'dialects' are often not even given the status of true languages, and are discouraged. I believe that developing countries must realize that their students will benefit from learning their own language well, and from learning their other school subjects in both their L1 and L2.

To turn to the second issue, that of appreciating Western mathematics, I do agree with my non-Western students that mathematical systems are culture-bound, and therefore that the term 'Western mathematics' is appropriate. But again I do not

believe that Western mathematics is absolutely impossible for non-Western students to understand. But they certainly have a case in stating that Western mathematics is foreign to their culture and hence difficult for them. They must acquire the entire analytical mindset, which may be very different from their traditional, holistic way of looking at the world. They must learn to distance themselves abstractly from subject matter rather than immersing themselves in it. Western mathematics is based on a cultural cognitive style which is not the same as the cognitive style of many peoples around the world.

As for the third issue, that Western mathematics is irrelevant, students in non-Western countries must realize that they are competing on a global scale with the developing countries, in economics and trade, in science and technology, and in the efficient use of manpower. If they wish to compete successfully, they must accept the fact that the game is being played according to Western rules, and that they must compete against the technology of the developed nations by learning the science and mathematics used in the world arena, namely Western mathematics.

The alternative, of course, is not to play the game at all, that is, to withdraw into cultural, economic, and scientific isolation from Western ideas. Some countries have attempted this option, for example, those Moslem countries which have turned towards fundamentalist values, but they all experience difficulty in competing with Western materialist values as 'enshrined' in television sets, flashy cars, and movie stars.

These are of course large political issues which are far beyond the scope of this book. But mathematics educators must be aware that education is very much tied to these political issues and cannot be treated in a vacuum. In any case, most educational decisions are taken by politicians anyway. For example, the decision concerning the language of instruction is usually taken for political reasons rather than for purely educational ones. And the hard facts of global politics today are that countries must educate scientists and mathematicians in the Western tradition if they hope to survive.

To my mind, the issue which is far more debatable than the necessity of training Western mathematicians is the question of what kind of mathematics should be taught to the vast majority of students who will never reach university level. In most of the developing countries I have observed, *all* students are guided onto the pre-university track, leading to abstract algebra and calculus, even though only a handful will succeed. Those who do not succeed

are given no practical mathematics instruction which will be relevant to their experience when they return to the village or take up work in a shop or factory. They may be able to compute the derivative of a complicated exponential or logarithmic function, but they are unable to estimate the amount of maize to plant in a given field, or to calculate compound interest on a loan.

I feel that special attention should be given to the 95% of students who need practical mathematical skills in daily life. This is where ethnomathematics comes into the picture, because children should be taught mathematics which is relevant to their specific culture and which is congruent with their language and their cultural value system.

Similar arguments can be applied to minority and lower-class students in developed countries. An elite few will proceed to university mathematics and will therefore need to learn in the formal academic language of universities. But the vast majority would benefit more from learning mathematics appropriate to their own environment, in language appropriate to their own culture.

Diversity of theories

This book has presented discussions of some of the major linguistic topics of concern to mathematics teachers around the world. It is now time to look back over the entire range of topics to see whether there are any common threads to the arguments presented in the various isolated chapters. I believe there are.

If there is a key word which sums up the results of the research on language and mathematics, surely diversity is the one. In theorizing about the way in which the human mind works, there is a tendency to try to explain everything by one theory. But there is no reason that the mind should work in only one way. The evidence is beginning to accumulate in support of a multifunctional mind, which can operate in one mode in some cases, and in other modes when the situation demands.

To claim that all thought, and hence all logic, is done in language, is one of those unitary theories. Why is it not possible that some thoughts are performed in language, while others are not? For example, if you were asked, "What is a corollary?" you would probably respond linguistically, giving an answer in words like, "A minor theorem which is the direct result of a major theorem." But if you were asked, "What is a helix?" you would

almost conjure up the picture of a helix in your mind, and you would probably 'define' it with a spiral motion of your hand.

The diversity of modes of thought is one of the marvelous things about the human mind, and makes it difficult to pin down thinking into narrow moulds. Moreover, the combination of thought modes working in parallel creates a synergistic effect by which we may treat complex notions. For example, think about the word 'triangle'. Probably you have called to mind a generic picture of a triangle. Is it isosceles, equilateral, or scalene? Actually, it was probably none of these until the words forced you to think about it. The concept was there in a fuzzy form which somehow represented all triangles. Now think about the word 'decagon'. This time you probably thought first about the word itself and said to yourself something like "Deca- means ten, so it is a ten-sided polygon." Your mind called up some kind of picture, but probably you could not actually count the ten sides. Thus the concept of decagon in your mind was both image and language, and the two worked together to produce something which could not be individually produced by either.

I would also guess that your image of a decagon was of a regular one, while your triangle might not have been equilateral. Why? Most likely because in geometry books the only decagons ever shown are regular, and if not, they are certainly convex. This is one example where mathematical language defines a concept one way ("Any ten-sided polygon") but the examples shown in mathematics classes form a set of images which yield a much restricted concept.

Thus it is virtually impossible to claim that the mind works in any one way. Even at any one instant the mind is likely to be operating in several modes.

Likewise, the question of which comes first, the thought or the language, is likely to have several answers. Even people who have no language are able to generalize and to solve problems. The perception of relations or similarities between objects is not confined to speakers of language, and has been observed in very young children. On the other hand, many concepts are defined using language. Similarities between objects may be noticed only because someone else has noticed the similarity and has given it a name.

For example, one mathematics teaching strategy is to show students several examples and to let them notice the similarity. The common property may then be given a name, although the concept has already been formed. Another strategy is to state a

word, define it, and then proceed to give examples. In the first example, the thought precedes the language; in the second, the language comes first.

The grandiose theories which attempt to explain all thought in just one way will probably not succeed, but may offer insights into mental processes. The strong Whorf hypothesis, for example, which claims that all thought is language, is no longer recognized as the all-encompassing answer it was once thought to be, but many of Whorf's brilliant writings reveal insights into how different cultures like the Hopi have radically different languages and radically different world-views from the Western ones. Whether there is a cause-effect relationship between the Hopi language and the Hopi world-view is not only not debated today, it is not even considered debatable, since language and culture are now considered so inextricably bound together. The question of which one is more basic is a chicken-and-egg question.

Similarly, the sociolinguistic theories of Bernstein cannot completely explain why lower-class students perform poorly in school and on standardized tests. The fact that lower-class children speak a different sort of 'code' may not be *the* reason they do not succeed in education, but it may be *one* of the many reasons. Bernstein's theory offers insight into the differences in thought and language between lower-class and middle-class children, even though it fails to establish a direct causal link to poor results at school.

The attempt of Logan to explain *everything* (mathematical logic, scientific method, monotheism, democracy, etc.) by the alphabet effect certainly cannot succeed entirely, but as in the case of Whorf and Bernstein, valuable insights are offered as to the relationship between the written word and thought. The accompanying question of right-brain versus left-brain thought has led to much exciting research, but a simple theory of just what functions are performed by which hemisphere of the brain remains beyond our grasp.

In a sense, it is satisfying that these unifying theories have failed to explain human thought completely. I see this failure as testimony to the complexity and adaptability of the human mind. In one situation it will work in one way, say spatially, while in other circumstances in will act linguistically, and in most situations it will act in many different ways concurrently. The great flexibility of human thought has educational consequences, namely, that diversity of experience is necessary in order to exercise the many types of thought.

Mathematical reality and psycholinguistic reality

One of the main themes of this book is that a wide range of experiences can produce positive, synergistic effects. Looking at life from two viewpoints produces more than the two viewpoints: it produces a different *way* of looking at things. That is the point of Cummins' threshold theory of bilingualism. Learning two or more languages produces an abstract language-awareness which may lead to a heightened ability to think abstractly in mathematics and science.

The notion of hierarchies of concepts is useful in this context. Looking at different aspects of objects - shape, colour, size, etc. - creates mental structures which make it easier to accommodate new knowledge and to scan problem-solving strategies.

We have seen that in many instances mathematical concepts are not the same as everyday concepts, and are not necessarily processed in the same way by the brain. If that is true, it raises a serious question for mathematics teachers: to what extent should we relate mathematics to reality in students' minds? It is possible that an attempted link between a mathematical word and its everyday equivalent may even be confusing to students.

I used to introduce the function concept to students by attempting to relate it to their real life experience of the common English word 'function'. My lecture/discussion would proceed something like the following:

> Suppose I say, "Your grade on the test is a function of how much you study." What do I mean?
>
> (*perhaps some discussion by students*)
>
> The number of hours you study exactly determines your grade. If you study 2 hours, you may get a C; 3 hours a B, etc. You might have a diagram that looks like this:

hours studied	*grade*
0	F
	D
1	
	C
2	B
	A
3	

If G (for grade) is the name of this grading function, I can write, for example, G(2.0 hours) = B, G(1.3 hours) = C, etc.

I can also write any combination of hour and grade as an ordered pair, such as (2.0, B), (1.3, C), etc. In fact, the set of all those ordered pairs will completely describe the grading function. I could have just as easily have defined the grading function G as that set of ordered pairs, which has the property that no two letters as second coordinates can come from the same number of hours as the first coordinate. It would be impossible for one person to have the pairs (2,B) and (2,C) - how could the person get both a B and a C on the same test after studying for 2 hours? On the other hand, (2,B) and (3,B) is possible - it would mean that whether you study 2 hours or 3 hours you will still get a B. So no two different pairs in the function may have the same first element, although two pairs may have the same second element.

I used to think this method was rather clever, since it started with a real life word and ended up with a mathematical definition for it. But I am not so sure any more. Logically, the mathematical function concept may be related to the use of the word in the sentence, "Your grade is a function of your study time", but psychologically it may not be. In the everyday sentence, the word 'function' may be a noun , but it is not a *thing* which you can see. Students feel they already 'know' what a function is in real life, but this new mathematical definition as a set of ordered pairs does

not coincide with their psychological reality. They may either be confused, or else feel that the teacher is misleading them by trying to convince them of a relationship of mathematics to common sense which does not really exist. Moreover, this is yet another example of language which may confuse students as to whether the teacher is using the mathematical register or everyday speech.

My own teaching experience has convinced me that mathematical logical equivalence is not identical to psycholinguistic equivalence. We tend to think, as mathematics teachers, that if A and B are members of an equivalence class, then either A or B may be substituted for each other in any and all circumstances. Psychologically this is not true, for A might be processed very differently from B. We have presented several examples of this phenomenon in Chapter 4. The statement 'If A then B' is logically equivalent to 'B or not A, as well as to 'If not B then not A', but it is very difficult for students to understand that the statements are logically the same, and they may not respond to the statements in the same way.

This is where I feel Chomsky (who was a mathematics major) falls down. He would say that there is a simple transformation from 'If A then B' to 'If not B then not A', and that the two statements have the same 'deep structure'. In fact, he is referring to deep formal-logical structure and not the deep psycholinguistic structure, since most people, even those fully 'competent' in the language, will not respond in the same manner to the two statements.

Nowhere does this difference between mathematical thought and psycholinguistic thought show up more clearly than in the use of the word 'or'. In mathematics it carries its inclusive meaning as representing the union of two sets, but psychologically it is often processed in its exclusive sense. This distinction between logical connectives as used in mathematics as opposed to their use in everyday life has been demonstrated by Jansson (1978):

> The field of mathematics uses logical connectives such as and, or, and if-then all the time. Where these are symbolized, as in formal logic, their meanings are clearly and unambiguously defined, and it is these meanings we wish to use in mathematics. However, we do not write mathematics in the symbols of formal logic, nor, at many levels, do we wish to. Thus the teacher and student of mathematics are going to use ordinary, natural language terminology. And herein lies

> the difficulty. The concepts associated with the natural language terminology are not congruent with those of the same terminology used in mathematics (or logic). (p. 186)

Parenthetically, the reader will no doubt realize that my use of the terms 'psycholinguistic thought' and 'psycholinguistic reality' tacitly expresses a bias towards the view that thought is strongly related (although not identical) to language, and that a person's reality is related to his or her culture, and hence to the language of that culture.

If mathematical reality is somewhat different from psycholinguistic reality, we must ask, "*Should* it be different?". For the ancient Greeks as well as for the Europeans of the Middle Ages, mathematics was a branch of philosophy, and as such was concerned with the analysis of real-life experience through crystallization into rational thought. For them, the question "What is a function?" would have been a question of isolating those relevant attributes of the already existing human concept of function. This spirit would therefore have led to a negative answer to the above question: no, mathematical reality should not be qualititatively different from psycholinguistic reality, but should abstract the essence of that reality. In fact, for them, such language as 'psycholinguistic reality' would have been unthinkable.

The philosophical method of abstracting certain properties is a typically Western notion. As we have seen in Chapter 11, other cultures find this practice rather odd if they are used to attacking problems in the opposite way: by entering *into* the object in question rather than by distancing oneself from it. Western mathematics is an extension of the philosophical, analytical method, but the change in the degree of analysis has led to a difference in the *nature* of the analysis.

This is a natural consequence. You cannot analyze a concept without altering it somewhat. If you scrutinize your intuitive concept of function in an analytical way, that concept will never be the same for you again. If you raise a subconscious thought to the level of consciousness, it can never again be a subconscious thought. So in this sense the mathematical reality, due to its analysis, will necessarily be somewhat different from the prior psycholinguistic reality. The danger comes when the process advances to such an altered state that the prior psycholinguistic reality and the mathematical reality are no longer even recognizable as being related.

Relations are similar to functions. The definition of a relation as a set of ordered pairs is at least within sight of the psychological act of relating things. But consider the relation ' < '. A pair (a,b) is an element of the 'less than' relation if and only if a is less than b. In mathematical language, we can write

$$(a,b) \in < \quad \text{iff} \quad a < b$$

The relation is well-defined, since any pair can immediately be said either to belong or not to belong to the relation. But psychologically, any relation presupposes a strength. 3 is less than 4, but 3 is less than 1,000,000 to a much greater degree. That strength is not seen in the above mathematical definition of relation - two objects are either related or not related, and no consideration is given to how strongly they are related. The intrinsic strength element of natural relationships has been sacrificed by Western mathematics in order to adhere to the zero-one nature of set definition.

Set theory was devised as a way of dealing abstractly with certain concepts. But in the simplification process, the inherently fuzzy nature of many real-life concepts has been lost. Psychologically, most sets are not mathematically well-defined. The set of qualified mathematics teachers, for example, is not a good mathematical set, because there are many degrees of qualification. One teacher may be more qualified than another, but who is to say whether qualification exists in an absolute sense - that A is qualified while B is not? Qualification is difficult to measure in a zero-one, all or nothing sense. Most concepts in our minds are similarly fuzzy.

The result of the set-theoretical view of mathematics is that new concepts are forced into the Procustean bed of set theory. The concepts of function and relation, for example, have been forced into this mould and have lost some of their psycholinguistic reality in the process.

I believe that the theory of fuzzy sets is a step in the right direction. This theory attempts to define a fuzzy set as a characteristic function which takes on values other than the traditional 0 and 1. An element can be partially a member of a fuzzy set, as defined by a number between 0 and 1. A teacher can be said to be, for instance, one-third qualified in the fuzzy sense. Unions and intersections of fuzzy sets are defined as maxima and minima of their characteristic functions.

Not only does fuzzy set theory mirror more closely the human fuzziness in concept formation, it also represents a return to functionalist thought as opposed to the reified or 'thing-ified' theories of traditional mathematics. Chapter 1 examined the structuralist-functionalist duality. The function concept is, *par excellence*, a functionalist notion, at least psychologically, but mathematics, with its preoccupation with structure, has turned it into a structuralist concept. Western mathematics has focused on the structure of functions and has not concentrated on how functions actually operate or interact with other objects. Evidence of this is the belated invention of the theory of distributions by Laurent Schwartz, well after physics had already been using Dirac's delta 'function' for years. Dirac had used this delta in the 'functional' sense of operating on or interacting with other functions. He defined the interaction of δ with an arbitrary test function f from the real numbers to the real numbers as

$$<\delta , f> = f(0)$$

This definition remained for a long time unacceptable to mathematicians, because such a 'function' would have to satisfy

$$\delta(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \text{infinity} & \text{for } x = 0. \end{cases}$$

It was only later, when Schwartz's distributions were defined by their actions on certain test functions that Dirac's delta could be treated formally as a linear functional on a space of test functions rather than as a function in its own right. Moreover, by applying the new definition (as linear functional) to other concepts, such as 'derivative', Schwartz's theory was able to define derivatives for discontinuous step functions.

This discussion may be becoming rather technical for the non-mathematical reader, but its essence should be clear: Dirac invented a concept which was useful to physics and was later found to have useful mathematical applications, but it took mathematicians a long time to accept his concept because it did not fall into their fixed structures of the time.

Historical mathematics - Husserl

One reason that mathematics has been able to move so far away from psycholinguistic reality is, I believe, that mathematics teachers have not stressed its history and development. Students are not given an appreciation that certain mathematical notions were invented in order to fill the need to describe how the human mind works. They are presented with the *products* of mathematics as a *fait accompli*, just as though, for example, set theory has existed forever outside the realm of human thought. It is given an existence of its own.

The idea of stressing the history and development of ideas has been explored by the twentieth century phenomenological philosopher Edmund Husserl (1970) in an essay entitled "The Origin of Geometry," in which he examines how language was used to bring the Greek geometers' ideas down to us today. His view of language is the Aristotelian theory whereby language is simply an encoder of pre-existing thoughts:

> The Pythagorean theorem, [indeed] all of geometry, exists only once, no matter how often or even in what language it may be expressed. It is identically the same in the "original language" of Euclid and in all "translations" (p. 357)

But the paradox of deductive mathematics is that in theory, the Pythagorean theorem exists *a priori* in each of us and has only to be realized. At some point, Pythagoras became aware of the theorem, and the theorem has been passed down to us today. So the origin of geometry is only an origin of its awareness. What must be passed on from generation to generation, language to language, is not the Pythagorean theorem *per se*, but the realization of it. For the mathematics teacher, to teach the theorem is actually to make students aware of something they somehow 'know' already. In order for students fully to comprehend the theorem, they must be able to repeat in their own minds Pythagoras' mental process of becoming aware of the theorem. This gives rise to the historical factor necessary, according to Husserl, in mathematics education.

> To understand geometry or any given cultural fact is to be conscious of its historicity. Making geometry self-evident, then, is the disclosure of its historical tradition.

> The ruling dogma of the separation in principle between epistomological elucidation and historical, even humanistic-psychological explanation, is fundamentally mistaken.

If the terms 'formal logical' and 'psycholinguistic' are substituted in the previous statement for 'epistomological' and 'humanistic-psychological', respectively, the statement becomes essentially what I have been saying in the previous pages.

Once again, the criticism of mathematics education is that it is product-oriented and answer-oriented, when it should be "a process, not a product" in Bruner's words. In a highly materialistic and technological world, of course, it is the products which matter, especially when those products can be used to make new products. Children are asked to accept ready-made material already packaged in the form of slick textbooks which have pre-digested the ideas into a sort of pabulum which must be taken as the absolute authority by the students simply because it is written in black-and-white. The authority of the written word is immense. Husserl understands this point:

> In human life, the originally intuitive life which creates its originally self-evident structures through activities on the basis of sense-experience very quickly and in increasing measure falls victim to the *seduction of language*. Greater and greater segments of this life lapse into a kind of talking and reading that is dominated purely by association. (p. 362)

> The possibility of the practical application of the derived laws, which was actually obvious in connection with the original developments, understandably led quickly, in the realm of praxis, to a habitually practiced method of using mathematics to bring about useful things. This method could naturally be handed down even without the ability for original self-evidence. Thus mathematics, emptied of meaning, could generally propagate itself. (p. 368)

Mathematics has been successful as a tool for technology, and so it is not surprising that it continues to be treated as a product. Success begets success. But mathematics, in its practical

success, has moved away from its psychological roots, as we have seen, and risks moving so far away that it is no longer traceable to its roots. It takes on an identity of its own, and begins to *control* thought rather than to reflect thought. We have seen from Ivan Illich (1988) how Western mathematics has begun to control language and to force it into the straitjacket of initials, Newspeak, and Uniquack. Husserl, too, sees a danger:

> Without the actually developed capacity for reactivating the original activities contained within its fundamental concepts, i.e., without the "what" and the "how" of its prescientific materials, geometry would be a tradition empty of meaning; and if we ourselves did not have this capacity, we could never even know whether geometry had or ever did have a genuine meaning, one that could really be 'cashed in'. (p. 366)

I am of course advocating the teaching of the history of mathematics in teacher education courses, so that teachers in turn may instill some of the sense of history and development in their students. But it should be equally clear from the foregoing that I am not in favour of a fact and date treatment of history. It does little good to force students to regurgitate such facts as who invented the calculus and when. It is more the 'how' and the 'why' which are important.

To return to my discussion of introducing the function concept to students, I believe that it is correct to show that such mathematical concepts have their roots in psycholinguistic reality. That is why I began my treatment with a common everyday use of the word 'function'. What worries me is that perhaps I am wrong in my reconstructed history of the origins of the mathematical function concept. I can only speculate as to its origins, and in presenting the link between the sentence "Your grade is a function of your study time" and the mathematical definition, I may be reconstructing history wrongly.

One other point concerning Husserl must be clarified. The above argument may appear to be saying that there is only one geometry, that invented by the Greeks and brought forward through history to today. This is not what Husserl is saying, although he *is* saying that the Greeks' geometry can be expressed in other languages. But other cultures may have their own geometries which may be quite different from that of the Greeks, and these geometries are no less valid than Euclid's. Husserl

recognizes that geometry is part of a cultural tradition, and as such is culture-specific. It is therefore necessary to ask whether the study of geometry is relevant for non-Western peoples.

Concluding remarks

The reader may have expected, upon opening this book, that its main theme would be that mathematics is essentially language and that most, if not all difficulties in mathematics learning could be traced to language difficulties. I hope I have not fallen into that trap. Indeed, in many cases I have tried to debunk theories which have attempted to argue this. We cannot escape the conclusion that some aspects of mathematics education are language-oriented, but some are not, and I have endeavoured to point out the extent to which they are or are not.

Thought, language, and culture are inextricably tied together. Perhaps it is our Western insistence on the analytical dissection into isolated units which is responsible for the failure to keep this notion in mind. Our experimental methodology of *ceteris paribus*, or "other things remaining equal", attempts to allow one factor to vary while keeping the others constant. It is impossible to keep, say, language and culture constant and to study thought in isolation, or to vary either language or culture while keeping the other constant.

Similarly, it is impossible for teachers to teach mathematics without teaching language and culture at the same time. This statement stops short of claiming that mathematics *is* language and culture; but teaching method, teacher-pupil interaction and problem-solving all rely so heavily on language and culture that mathematics teachers cannot help conveying linguistic and cultural information.

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This book is designed for those interested in the teaching of mathematics, in both first language and second language contexts, and is directed at mathematics teachers and also educators in the field of language for special purposes. It is designed to acquaint these people with the main issues of language in mathematics teaching.

The book gives an overview which introduces teachers to the role which language plays in the formation of mathematical concepts, in the use of formal logical inference, and in the social structure of the classroom. It uses concepts from many academic disciplines: education, psychology, mathematics, sociology and, of course, linguistics.

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